

Research Article

A Study of the Wave Dynamics for the Reaction-Diffusion Brusselator System and RKL Equation

Onur Alp İlhan ¹ and Jalil Manafian ^{2,3,4}

¹Department of Mathematics, Faculty of Education, Erciyes University, Melikgazi, Kayseri 38039, Turkey

²Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

³Natural Sciences Faculty, Lankaran State University, 50, H. Aslanov Str., Lankaran, Azerbaijan

⁴Faculty of Physics and Mathematics, Department of General Mathematics, Nakhchivan State University, Nakhchivan, Azerbaijan

Correspondence should be addressed to Onur Alp İlhan; oailhan@erciyes.edu.tr and Jalil Manafian; j_manafianheris@tabrizu.ac.ir

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In this article, the reaction-diffusion Brusselator system and the Radhakrishnan, Kundu, and Laskshmanan (RKL) equation are discussed. The investigation is focused on the solution procedure of the nonlinear evolution equations in chemical and biological processes. Two approaches, namely the $\tan(\theta/2)$ -expansion approach and the $\exp(-\Omega(\eta))$ -expansion approach, are considered. A multitude of precise solutions is derived for the aforementioned equation. The phase portraits are generated using the Maple program by specifying predefined parameters. The solutions of the stated equations include the $\tan$ -expansion function solutions, the kink solitary wave solutions, and the solitary wave solutions. The obtained results using both methods are useful and constructive and also are reliable for solving nonlinear partial differential equations used in nonlinear sciences.

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1. Introduction

In recent times, there has been a pervasive adoption of intricate nonlinear partial differential equations (NLPDEs) across diverse fields including nonlinear optics, fluid mechanics, quantum physics, biology, communication, and control. Prominent instances of these equations encompass the Schrödinger equation, the Radhakrishnan–Kundu–Laskshmanan (RKL) equation, the Kundu–Mukherjee–Naskar equation, the Laskshmanan–Porsezian–Daniel equation, the Triki–Biswas equation, the Fokas–Lenells equation, the Gerdjikov–Ivanov equation, the Ginzburg–Landau equation, and the Biswas–Arshed equation. The analysis of dynamic phenomena and accurate solutions related to the transmission of waves in the field of academic

research has constantly shown a substantial interest in the study of complex NLPDEs [1–4].

In this article, we consider the following reaction-diffusion (RD) Brusselator system [5, 6] as

$$\begin{aligned}\frac{\partial u}{\partial t} &= B + u^2 v - (A + 1)u + \mu \nabla^2 u, \\ \frac{\partial v}{\partial t} &= Au - u^2 v + \mu \nabla^2 v,\end{aligned}\tag{1}$$

in which $u(x, y, t)$ and $v(x, y, t)$ indicate chemical concentrations of intermediate reaction products and A and B are constant concentrations of input reagents and ∇^2 is Laplace operator. Also, μ (diffusion coefficient) is a constant. It is well

known that for small values of the diffusion coefficient μ , the steady state solution of Brusselator system (16) converges to equilibrium point $(B, A/B)$, if $1 - A + B^2 \geq 0$. RD systems are often seen in the investigation of chemical and biological processes and are inherently represented by partial differential equations. The system denoted by equation (16) represents a biochemical model. In this study, we examine the RD Brusselator model, which represents a chemical RD process using the $\tan(\phi/2)$ -expansion approach. In a study conducted by Khan et al. [5], the authors examined the use of Bernstein polynomials in obtaining numerical solutions for the nonlinear fractional-order Brusselator system. Additionally, Wazwaz [7] employed the Adomian decomposition approach to solve the RD Brusselator model. Chowdhury and colleagues [6] used the homotopy perturbation technique as a computational approach to address the RD Brusselator system. Additionally, Jafari et al. [8] used the variational iteration approach to examine the approximate solutions of the aforementioned system. The precise solutions of the Brusselator RD model were achieved using the Exp-function approach as documented in the study of [9]. The Brusselator reaction model has significant significance in both the fields of biology and chemistry. Since its proposal by Prigogine and Lefever in 1968, the model has garnered significant interest and has been extensively studied by several researchers using various methodologies [10–12] to investigate its features. In their study, Yan and Zhang used the Darboux transformation technique to get precise solutions for the Brusselator RD model, as documented in [13]. The RKL equation is given as

$$iu_x + u_{tt} + \frac{\gamma_2}{3\gamma_1}|u|^2u + i\gamma_1u_{ttt} + i\gamma_2(|u|^2u)_t = 0, \quad (2)$$

where γ_1 and γ_2 are parameters of equation (2). In [14], many precise solutions were derived for two specific types of RKL models. The investigation of the Exp-function approach for solving nonlinear RKL models has been conducted by Ganji et al. [15]. Several effective methods for obtaining the exact solutions to nonlinear models are presented and refined including the direct method in soliton theory [16], solving nonlinear fractional partial differential equations [17], solution of the linear fractional PDEs [18], variational iteration method [19], application of semi-analytic methods [20], homotopy perturbation method [21], Laplace Adomian decomposition method [22], travelling wave solutions [23], solitary wave solutions [24], (G'/G) -expansion method [25, 26], the extended tanh method [27], the modified Zakharov–Kuznetsov equation [28], and nonlinear ultrasonic waves propagation in fluids [29]. The exact solutions of the fractional Hirota–Satsuma coupled KdV equation in the Beta fractional derivative were obtained [30]. The $(2+1)$ -dimensional Kadomtsev–Petviashvili–Sawada–Kotera–Ramani equation was investigated by making use of Hirota's bilinear form to obtain soliton, breather, and lump solutions [31]. Each approach must meet specific criteria to become part of the governing framework. In recent times, there has been a growing inclination among researchers and analysts to obtain precise soliton solutions

for dynamical systems. The accessibility of computer tools that simplify the laborious and time-intensive mathematical computations involved in this process facilitates this undertaking [32–35].

Some nonlinear problems were investigated in variety of applications in chemical analysis and nonlinear engineering [36–44]. A high-order meshless symplectic algorithm for Hamiltonian wave equation was suggested by using highly accurate radial basis functions (RBFs) quasi-interpolation operator in [45]. Three-dimensional seepage spring element method using the element flow matrix in finite element analysis was used for pore seepage problems [46]. The efficient multiscale squeeze-and-excitation network, spatial graph convolutional network, and gated recurrent unit-temporal convolutional network modules were designed for the multiscale spatial-temporal interaction fusion network model [47]. The electromagnetic-inverse-scattering problem was solved by a novel two-step deep-learning approach [48]. Small parameters analysis was applied to two potential issues for Maxwell–Schrödinger system in [49]. The discontinuous Galerkin time-domain method was adopted for electromagnetic simulation and the time-domain finite element method was utilized for thermal simulation in [50]. A fast wideband electromagnetic scattering analysis method based on Taylor expansion and higher order hierarchical vector basis functions extracting a phase term from the green function [51]. A novel transient field-circuit simulation method based on the time-domain integral equation and circuit macromodels was investigated in [52].

The primary objective of this work is to investigate the soliton and traveling wave solutions of the aforementioned equations via the use of the traveling wave theory and the $\tan(\theta/2)$ -expansion approach. Furthermore, the equations presented in the study have been successfully solved to give solitary wave solutions. The present article is structured in the following manner: The technique of $\tan(\theta/2)$ -expansion is presented in Section 2. Furthermore, the traveling wave solutions of equations (1) and (2) are derived using the aforementioned approach for two nonlinear models discussed in Section 3. The study presented in Sections 4 and 5 focuses on the $\exp(-\Omega(\eta))$ -expansion approach and its applications in the analysis of two nonlinear equations. The conclusion is delivered in Section 6.

2. Depiction of $\tan(\theta/2)$ -Expansion Approach

Step 1. Consider the nonlinear PDE form as

$$\mathcal{P}(u, u_x, u_t, u_{xx}, u_{tt}, \dots) = 0. \quad (3)$$

Where has changed to an ODE

$$\mathcal{S}(u, u', -\mu u', u'', \mu^2 u'', \dots) = 0. \quad (4)$$

By using $\xi = x - \mu t$, as the wave variable and moreover k, m and n are free constants.

Step 2. The closed form will be as follows:

$$u(\xi) = U(\Phi) = \sum_{k=0}^r A_k \tan\left(\frac{\theta}{2}\right)^k + \sum_{k=1}^m B_k \cot\left(\frac{\theta}{2}\right)^k. \quad (5)$$

Where

$$\begin{cases} A_k (0 \leq k \leq r), \\ B_k (1 \leq k \leq m), \\ A_r \neq 0, B_m \neq 0, \theta = \theta(\xi). \end{cases} \quad (6)$$

Where satisfies in the below ODE:

$$\theta' = a \sin(\theta) + b \cos(\theta) + c. \quad (7)$$

The related solutions to (7) will be as

Family 1: When $a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$,

$$\theta(\xi) = -2 \tan^{-1} \left[\frac{a}{b-c} + \frac{\sqrt{c^2 - b^2 - a^2}}{b-c} \tan \left(\frac{\sqrt{c^2 - b^2 - a^2}}{2} (\xi + C) \right) \right]. \quad (8)$$

Family 2: When $a^2 + b^2 - c^2 > 0$ and $b - c \neq 0$,

$$\theta(\xi) = -2 \tan^{-1} \left[\frac{a}{b-c} - \frac{\sqrt{b^2 + a^2 - c^2}}{b-c} \tanh \left(\frac{\sqrt{b^2 + a^2 - c^2}}{2} (\xi + C) \right) \right]. \quad (9)$$

Family 3: When $a^2 + b^2 - c^2 > 0$, $b \neq 0$ and $c = 0$,

$$\theta(\xi) = 2 \tan^{-1} \left[\frac{a}{b} + \frac{\sqrt{b^2 + a^2}}{b} \tanh \left(\frac{\sqrt{b^2 + a^2}}{2} (\xi + C) \right) \right]. \quad (10)$$

Family 4: When $a^2 + b^2 - c^2 < 0$, $c \neq 0$ and $b = 0$,

$$\theta(\xi) = 2 \tan^{-1} \left[-\frac{a}{c} + \frac{\sqrt{c^2 - a^2}}{c} \tan \left(\frac{\sqrt{c^2 - a^2}}{2} (\xi + C) \right) \right]. \quad (11)$$

Family 5: When $a^2 + b^2 - c^2 > 0$, $b - c \neq 0$ and $a = 0$,

$$\theta(\xi) = 2 \tan^{-1} \left[\sqrt{\frac{b+c}{b-c}} \tanh \left(\frac{\sqrt{b^2 - c^2}}{2} (\xi + C) \right) \right]. \quad (12)$$

Family 6: When $a = 0$ and $c = 0$,

$$\theta(\xi) = \tan^{-1} \left[\frac{e^{2b(\xi+C)} - 1}{e^{2b(\xi+C)} + 1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)} + 1} \right]. \quad (13)$$

Family 7: When $b = 0$ and $c = 0$,

$$\theta(\xi) = \tan^{-1} \left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)} + 1}, \frac{e^{2a(\xi+C)} - 1}{e^{2a(\xi+C)} + 1} \right]. \quad (14)$$

Family 8: When $a^2 + b^2 = c^2$,

$$\theta(\xi) = -2 \tan^{-1} \left[\frac{(b+c)(a(\xi+C)+2)}{a^2(\xi+C)} \right]. \quad (15)$$

Family 9: With $a = c$, then we get $\theta(\xi) = 2 \tan^{-1} [((b+c)e^{b(\xi+C)} + 1)/((b-c)e^{b(\xi+C)} - 1)]$.

Family 10: With $c = a$, then we get $\theta(\xi) = 2 \tan^{-1} [((a+b)e^{b(\xi+C)} - 1)/(1 - (a-b)e^{b(\xi+C)})]$.

Family 11: With $b = 0, c = a$, then we get $\theta(\xi) = 2 \tan^{-1} [(a(\xi+C)+2)/(a(\xi+C))]$.

Family 12: With $b = 0, c = 0$, then we get $\theta(\xi) = \tan^{-1} (2(e^{a(\xi+C)}/(e^{2a(\xi+C)} + 1)), - (e^{2a(\xi+C)} - 1)/(e^{2a(\xi+C)} + 1))$.

Family 13: With $b = c$, then we get $\theta(\xi) = 2 \tan^{-1} ((e^{a(\xi+C)} - c)/a)$.

Family 14: With $b = -c$, then we get $\theta(\xi) = 2 \tan^{-1} ((ae^{a(\xi+C)})/(1 - ce^{a(\xi+C)}))$,

In which $A_k (k = 0, 2, \dots, r), B_k (k = 1, 2, \dots, m), a, b$ and c are free values. For finding the value r use the balance principle.

Step 3. Solving the got algebraic equations and then replacing $A_0, A_1, B_1, \dots, A_r, B_r, \mu$ in (5), the exact solutions according to parameters will be obtained.

3. Two Nonlinear Models

3.1. The 2-Dimensional RD Brusselator System. Consider the 2-dimensional RD Brusselator system as

$$\begin{aligned} \frac{\partial u}{\partial t} &= B + u^2 v - (A+1)u + \mu \nabla^2 u, \\ \frac{\partial v}{\partial t} &= Au - u^2 v + \mu \nabla^2 v, \end{aligned} \quad (16)$$

in which $u(x, y, t)$ and $v(x, y, t)$ indicate the chemical concentrations of intermediate reaction products and A and B are constant concentrations of input reagents and ∇^2 is Laplace operator. Also μ (diffusion coefficient) is a constant. It is well known that for small values of the diffusion coefficient μ , the steady state solution of Brusselator system (16) converges to equilibrium point $(B, A/B)$, if $1 - A + B^2 \geq 0$. Hence, by utilizing the below transformation

$$v = u^{-1}, \quad (17)$$

system (16) transformed

$$\begin{aligned} \frac{\partial u}{\partial t} &= B + u - (A + 1)u + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\ \frac{\partial u^{-1}}{\partial t} &= Au - u + \mu \left(\frac{\partial^2 u^{-1}}{\partial x^2} + \frac{\partial^2 u^{-1}}{\partial y^2} \right). \end{aligned} \quad (18)$$

Employing the transformation $\xi = x + y + \eta t$, equation (18) changes to

$$\begin{aligned} \eta u' &= u - Au - u - 2\mu u'' + B, \\ -\eta \frac{u'}{u^2} &= -u + Au - 2\mu \frac{uu'' - 2u'^2}{u^3}, \end{aligned} \quad (19)$$

by simplifying the above system, one get

$$\eta u' = -Au - 2\mu u'' + B, \quad (20)$$

$$-\eta uu' = (A - 1)u^4 - 2\mu(uu'' - 2u'^2), \quad (21)$$

by inserting (20) into (21) the following equation will be concluded as

$$Au^2 - Bu = (A - 1)u^4 + 4\mu u'^2. \quad (22)$$

By balancing u'^2 and u^4 gets

$$\begin{aligned} A = 0, B = 0, a &= \frac{2A_0cB_1}{A_0^2 + B_1^2}, b = -\frac{c(A_0^2 - B_1^2)}{A_0^2 + B_1^2}, \mu = \frac{(A_0^2 + B_1^2)^2}{4B_1^2c^2}, A_0 = A_0, A_1 = 0, B_1 = B_1, \\ u(\xi) &= A_0 + B_1 \cot\left(\frac{\theta}{2}\right), \end{aligned} \quad (28)$$

in which A_1, k and B_1 are free values. By utilizing equation (28) and Family 8, one become

$$2r + 2 = 4r, \quad \Rightarrow r = 1. \quad (23)$$

Then, the analytical solution will as

$$u(\xi) = A_0 + A_1 \tan\left(\frac{\theta}{2}\right) + B_1 \cot\left(\frac{\theta}{2}\right). \quad (24)$$

Appending (7) and (24) into (19), we receive the following sets of nontrivial solutions.

3.1.1. Set I

$$A = 1, B = 0, b = -c, \mu = \frac{1}{4a^2}, A_0 = \frac{cB_1}{a}, A_1 = 0, B_1 = B_1, \quad (25)$$

$$u(\xi) = \frac{cB_1}{a} + B_1 \cot\left(\frac{\theta}{2}\right),$$

in which a, c, k and B_1 are free values. By utilizing the (25) and Family 14, one become

$$u_1(\xi) = \frac{cB_1}{a} + B_1 \frac{1 - ce^{a(\xi+C)}}{ae^{a(\xi+C)}}, \quad \xi = x + y + \eta t, \quad (26)$$

$$v_1(\xi) = \left[\frac{cB_1}{a} + B_1 \frac{1 - ce^{a(\xi+C)}}{ae^{a(\xi+C)}} \right]^{-1}, \quad \xi = x + y + \eta t. \quad (27)$$

From Figure 1, it can be seen that the kink soliton has one ascending mode in the (x, t) -plane, which is represented by a 3D plot, density plot, and 2D plot. Also, from Figure 2, it can be discovered that the kink soliton solution has made in the (x, t) -plane, which is represented by a 3D plot, density plot, and 2D plot.

3.1.2. Set II

$$u_1(\xi) = A_0 - B_1 \left[\frac{((2A_0cB_1)/(A_0^2 + B_1^2))^2 (\xi + C)}{(-(c(A_0^2 - B_1^2))/(A_0^2 + B_1^2)) + c((2A_0cB_1)/(A_0^2 + B_1^2))(\xi + C) + 2)} \right], \quad \xi = x + y + \eta t, \quad (29)$$

$$v_1(\xi) = \left[A_0 - B_1 \left[\frac{((2A_0cB_1)/(A_0^2 + B_1^2))^2 (\xi + C)}{(-(c(A_0^2 - B_1^2))/(A_0^2 + B_1^2)) + c((2A_0cB_1)/(A_0^2 + B_1^2))(\xi + C) + 2)} \right] \right]^{-1}, \quad \xi = x + y + \eta t. \quad (30)$$

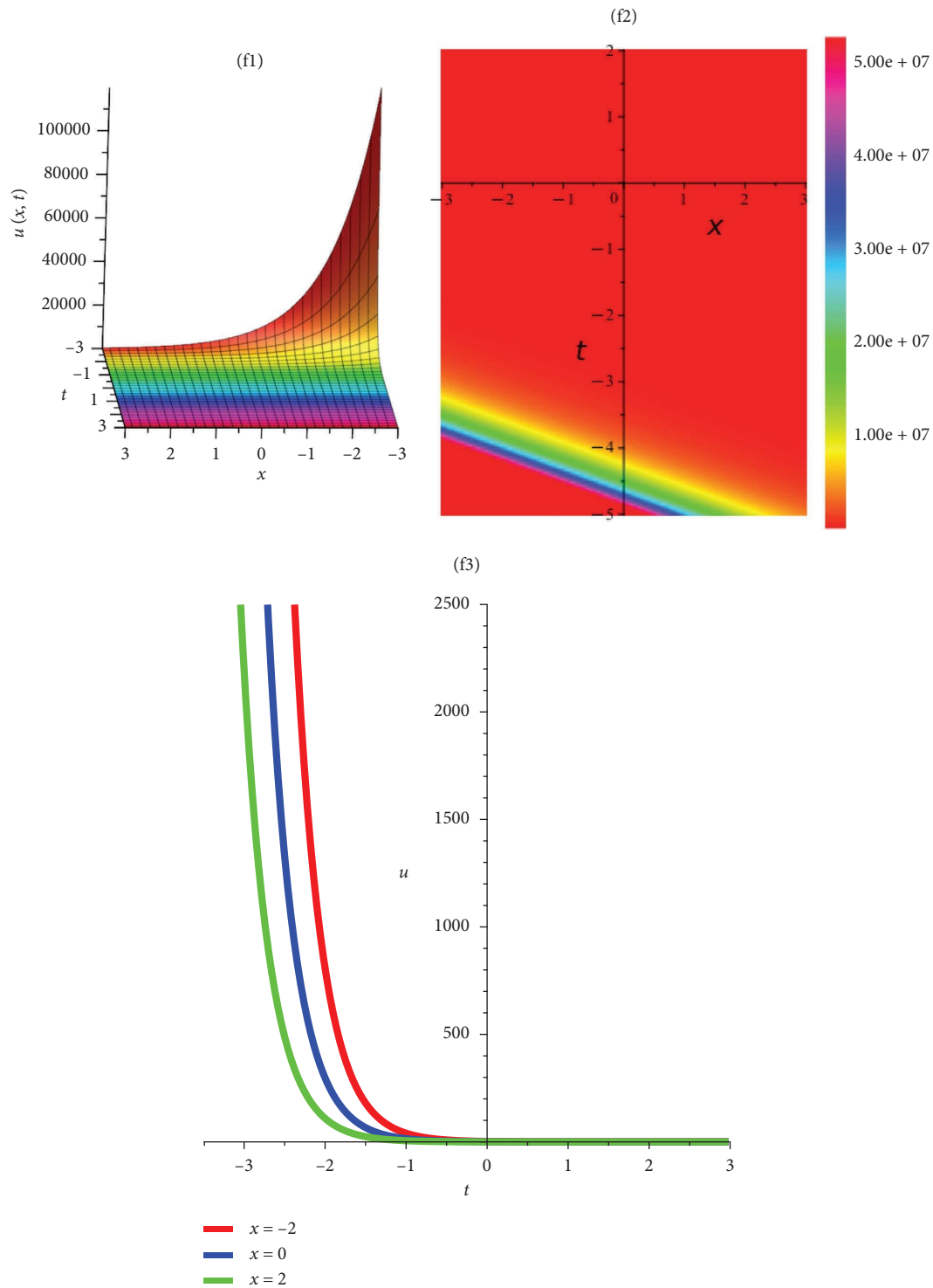


FIGURE 1: Graph of kink soliton solution $u_1(x, y, t)$ (26) for $A = 1, a = 1, c = 4$, and $y = 1$.

3.1.3. Set III

$$A = 1, B = 0, a = a, b = c, c = c, \mu = \frac{1}{4a^2}, A_0 = \frac{A_1 c}{a}, A_1 = A_1, B_1 = 0,$$

$$u(\xi) = \frac{A_1 c}{a} + A_1 \tan\left(\frac{\theta}{2}\right), \quad (31)$$

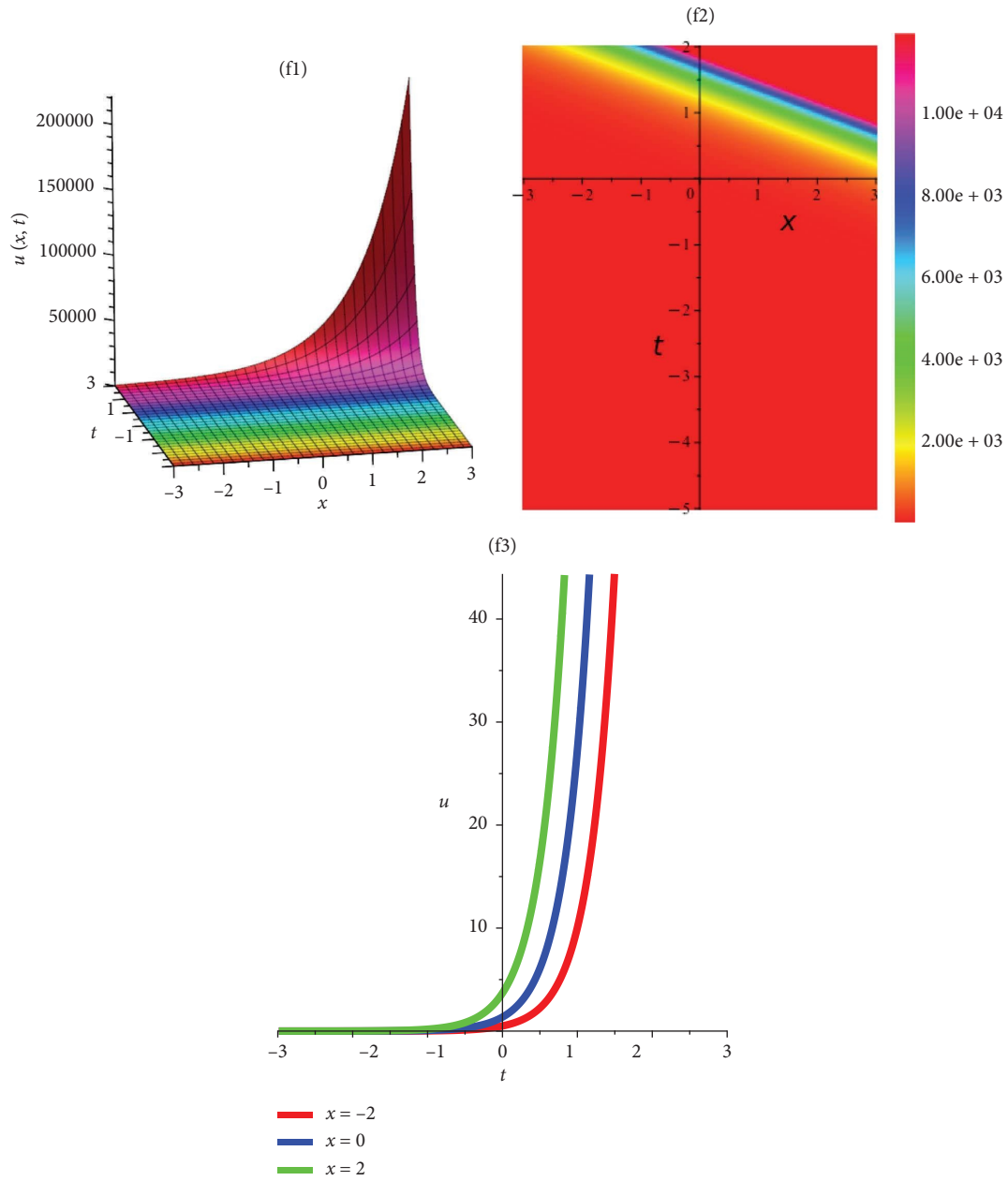


FIGURE 2: Graph of kink soliton solution $v_1(x, y, t)$ (27) for $A = 1, a = 4, c = 3$, and $y = 1$.

in which a, c and A_1 are free values. By utilizing equation (31) and Family 13, one become

$$u_1(\xi) = \frac{A_1 c}{a} + A_1 \left(\frac{e^{a(\xi+C)} - c}{a} \right), \quad \xi = x + y + \eta t, \quad (32)$$

$$v_1(\xi) = \left[\frac{A_1 c}{a} + A_1 \left(\frac{e^{a(\xi+C)} - c}{a} \right) \right]^{-1}, \quad \xi = x + y + \eta t. \quad (33)$$

3.1.4. Set IV

$$A = 0, B = 0, a = \frac{2A_0cA_1}{A_0^2 + A_1^2}, b = \frac{c(A_0^2 - A_1^2)}{A_0^2 + A_1^2}, c = c, \mu = \frac{(A_0^2 + A_1^2)^2}{4A_1^2c^2}, A_0 = A_0, A_1 = A_1, B_1 = 0, \quad (34)$$

$$u(\xi) = A_0 + A_1 \tan\left(\frac{\theta}{2}\right),$$

in which a, c and A_1 are free values. By utilizing equation (34) and Family 8, one become

$$u_1(\xi) = A_0 - A_1 \left[\frac{\left(\left(\left(c(A_0^2 - A_1^2) \right) / (A_0^2 + A_1^2) \right) + c \right) \left(\left((2A_0cA_1) / (A_0^2 + A_1^2) \right) (\xi + C) + 2 \right)}{\left((2A_0cA_1) / (A_0^2 + A_1^2) \right)^2 (\xi + C)} \right], \quad \xi = x + y + \eta t, \quad (35)$$

$$v_1(\xi) = \left\{ A_0 - A_1 \left[\frac{\left(\left(\left(c(A_0^2 - A_1^2) \right) / (A_0^2 + A_1^2) \right) + c \right) \left(\left((2A_0cA_1) / (A_0^2 + A_1^2) \right) (\xi + C) + 2 \right)}{\left((2A_0cA_1) / (A_0^2 + A_1^2) \right)^2 (\xi + C)} \right] \right\}^{-1}, \quad \xi = x + y + \eta t. \quad (36)$$

3.1.5. Set V

$$A = \frac{4(b+c)A_1^2}{4bA_1^2 + 4cA_1^2 + b - c}, B = 0, a = 0, b = b, c = c, \mu = \frac{A_1^2}{(b-c)(4bA_1^2 + 4cA_1^2 + b - c)}, A_0 = 0, \quad (37)$$

$$A_1 = A_1, B_1 = -\frac{(b+c)A_1}{b-c}, (\xi) = A_1 \tan\left(\frac{\theta}{2}\right) - \frac{(b+c)A_1}{b-c} \cot\left(\frac{\theta}{2}\right),$$

in which b, c and A_1 are free values. By utilizing equation (37) and Family 5, one become

$$u_1(\xi) = A_1 \left[\sqrt{\frac{b+c}{b-c}} \tanh\left(\frac{\sqrt{b^2 - c^2}}{2} (\xi + C)\right) \right] - \frac{(b+c)A_1}{b-c} \left[\sqrt{\frac{b-c}{b+c}} \coth\left(\frac{\sqrt{b^2 - c^2}}{2} (\xi + C)\right) \right], \quad \xi = x + y + \eta t, \quad (38)$$

$$v_1(\xi) = \left\{ A_1 \left[\sqrt{\frac{b+c}{b-c}} \tanh\left(\frac{\sqrt{b^2 - c^2}}{2} (\xi + C)\right) \right] - \frac{(b+c)A_1}{b-c} \left[\sqrt{\frac{b-c}{b+c}} \coth\left(\frac{\sqrt{b^2 - c^2}}{2} (\xi + C)\right) \right] \right\}^{-1}, \quad \xi = x + y + \eta t, \quad (39)$$

with the below parameters

$$\begin{cases} b = \frac{1}{2} \sqrt{\frac{A}{\mu}}, & c = 0, & A_1 = \frac{1}{2} \sqrt{\frac{A}{1-A}}, \\ b = -\frac{4AA_1^2 - 4A_1^2 - A}{8A_1\sqrt{\mu(1-A)}}, & c = \frac{4AA_1^2 - 4A_1^2 + A}{8A_1\sqrt{\mu(1-A)}}, & A_1 = A_1. \end{cases} \quad (40)$$

3.1.6. Set VI

$$A = 1, B = 2A_0, a = a, b = 0, c = 0, \mu = \frac{1}{4a^2}, A_0 = A_0, \quad (41)$$

$$A_1 = \frac{A_0^2}{4B_1}, B_1 = B_1, (\xi) = A_0 + \frac{A_0^2}{4B_1} \tan\left(\frac{\theta}{2}\right) + B_1 \cot\left(\frac{\theta}{2}\right),$$

in which a, A_0 and B_1 are free values. By utilizing equation (41) and Family 7, one become

$$u_1(\xi) = A_0 + \frac{A_0^2}{4B_1} \tan\left(\frac{1}{2} \tan^{-1} \left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)} + 1}, \frac{e^{2a(\xi+C)} - 1}{e^{2a(\xi+C)} + 1} \right]\right) + B_1 \cot\left(\frac{1}{2} \tan^{-1} \left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)} + 1}, \frac{e^{2a(\xi+C)} - 1}{e^{2a(\xi+C)} + 1} \right]\right), \quad (42)$$

$$v_1(\xi) = \left\{ A_0 + \frac{A_0^2}{4B_1} \tan\left(\frac{1}{2} \tan^{-1} \left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)} + 1}, \frac{e^{2a(\xi+C)} - 1}{e^{2a(\xi+C)} + 1} \right]\right) + B_1 \cot\left(\frac{1}{2} \tan^{-1} \left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)} + 1}, \frac{e^{2a(\xi+C)} - 1}{e^{2a(\xi+C)} + 1} \right]\right) \right\}^{-1}, \quad (43)$$

$$\xi = x + y + \eta t.$$

3.1.7. Set VII

$$A = \frac{3A_0^2}{3A_0^2 - 1}, B = -\frac{2A_0^3}{3A_0^2 - 1}, a = \frac{8A_0cA_1}{A_0^2 + 4A_1^2}, b = \frac{c(A_0^2 - 4A_1^2)}{A_0^2 + 4A_1^2}, c = c, \quad (44)$$

$$\mu = \frac{(A_0^2 + 4A_1^2)^2}{64A_1^2c^2(3A_0^2 - 1)},$$

$$A_0 = A_0, A_1 = A_1, B_1 = \frac{A_0^2}{4A_1}, (\xi) = A_0 + A_1 \tan\left(\frac{\theta}{2}\right) + \frac{A_0^2}{4A_1} \cot\left(\frac{\theta}{2}\right),$$

in which a, A_0 and A_1 are free values. By utilizing equation (44) and Family 2, one become

$$u_1(\xi) = A_0 + A_1 \left[\frac{a}{b-c} + \frac{\sqrt{b^2 + a^2 - c^2}}{b-c} \tanh\left(\frac{\sqrt{b^2 + a^2 - c^2}}{2} (\xi + C)\right) \right] + \frac{A_0^2}{4A_1} \left[\frac{a}{b-c} + \frac{\sqrt{b^2 + a^2 - c^2}}{b-c} \tanh\left(\frac{\sqrt{b^2 + a^2 - c^2}}{2} (\xi + C)\right) \right]^{-1}, \quad (45)$$

$$v_1(\xi) = \left\{ A_0 + A_1 \left[\frac{a}{b-c} + \frac{\sqrt{b^2 + a^2 - c^2}}{b-c} \tanh\left(\frac{\sqrt{b^2 + a^2 - c^2}}{2} (\xi + C)\right) \right] + \frac{A_0^2}{4A_1} \left[\frac{a}{b-c} + \frac{\sqrt{b^2 + a^2 - c^2}}{b-c} \tanh\left(\frac{\sqrt{b^2 + a^2 - c^2}}{2} (\xi + C)\right) \right]^{-1} \right\}^{-1}, \quad \xi = x + y + \eta t, \quad (46)$$

$$b^2 + a^2 - c^2 = \frac{48A_0^2c^2A_1^2}{(A_0^2 + 4A_1^2)^2} > 0.$$

Figure 3 is presented so that the soliton has one ascending mode in the (x, t) -plane, which is represented by a 3D plot, density plot, and 2D plot. Also, using Figure 4 one

can understand that the soliton solution has made in the (x, t) -plane, which is represented by a 3D plot, density plot, and 2D plot.

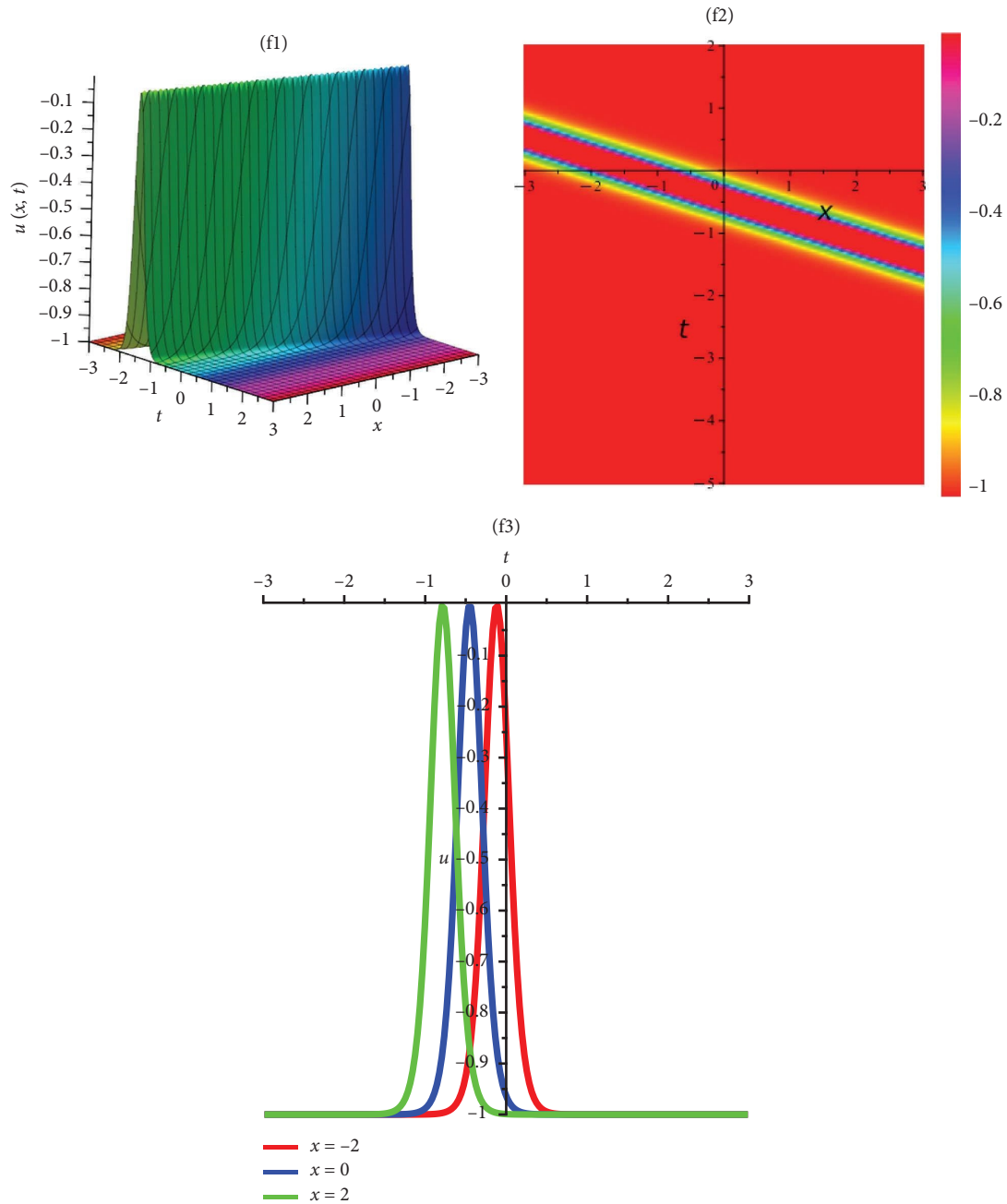


FIGURE 3: Graph of kink wave solution $u_1(x, y, t)$ (45) for $A = 1.5, B = -1, A_0 = 1, A_1 = 2, c = 4, \eta = 3, \mu = -289/8192$ and $y = 1$.

3.2. *The RKL Equation.* We next consider the nonlinear RKL equation as follows:

$$i u_x + u_{tt} + \frac{\gamma_2}{3\gamma_1} |u|^2 u + i\gamma_1 u_{ttt} + i\gamma_2 (|u|^2 u)_t = 0. \quad (47)$$

Therefore, by using the following transformations

$$U(\xi) = e^{i\theta} v(\xi), \quad \xi = \alpha x + \omega t, \quad \theta = kx - mt, \quad t^2 = -1, \quad (48)$$

Equation (47) transforms to an ODE as follows:

$$\begin{aligned} & i \left((\alpha + 2m\omega + 3m^2\omega\gamma_1) v' - 3\gamma_2 \omega v'^{\nu^2} - \gamma_1 \omega^3 v''' \right) - (k + m^2 + m^3\gamma_1) v \\ & + \gamma_2 \left(\frac{1}{3\gamma_1} + m \right) v^3 + \omega^2 (3\gamma_1 m + 1) v'' = 0. \end{aligned} \quad (49)$$

Balancing the v'' and v^3 by using the homogenous principal, we get

$$R + 2 = 3r \Rightarrow r = 1. \quad (50)$$

Then, the analytical solution is as below form

$$V(\xi) = A_0 + A_1 \tan\left(\frac{\theta}{2}\right) + B_1 \cot\left(\frac{\theta}{2}\right). \quad (51)$$

Replacing (7) and (51) into (49), we get the following sets of nontrivial solutions.

3.2.1. Set I

$$\begin{aligned} \omega &= \frac{3\gamma_1 m + 1}{\gamma_1} d, A_0 = \mp \frac{d(-6\gamma_1 m i - 2i + 9ad + 27ad\gamma_1 m)}{18\gamma_1 \gamma_2 d(-2\gamma_1 \gamma_2)^{-(1/2)}}, \quad A_1 = 0, B_1 = \pm \frac{d}{\sqrt{-2\gamma_1 \gamma_2}}, \quad m = m, \\ d &= \pm \frac{1}{9bc(b+c)} \sqrt{b(b+c) \left[\mp 3(a^2 + b^2 - c^2) + \sqrt{9(a^2 + b^2)^2 + (9c^4 - 18a^2 c^2 - 42b^2 c^2 - 24bc^3)} \right]}, \\ \alpha &= \frac{(3\gamma_1 m + 1)^3 da^4}{27bc^2(b+c)\gamma_1^2} + \frac{1}{2187bc^2(b+c)\gamma_1^2 d} \left\{ [(3\gamma_1 m + 1)^3 a^2 [162(b^2 - c^2) + 4]] + (81(3\gamma_1 m + 1)^3 d^2 b^4 + (-243(3\gamma_1 m + 1)(9\gamma_1^2 m^2 + 6\gamma_1 m - 2)d^2 c^2 \right. \\ &\quad \left. + 4(3\gamma_1 m + 1)^3 b^2 - 81(3\gamma_1 m + 4)(3\gamma_1 m - 2)(3\gamma_1 m + 1)d^2 c^3 b + 81(3\gamma_1 m + 1)^3 d^2 c^4 - 4(3\gamma_1 m + 1)^3 c^2) \right\}, \\ K &= \frac{1}{6561bc^2(b+c)\gamma_1 d^2} \left\{ [81(3\gamma_1 m + 1)^3 b^4 + (-6561m^2 \gamma_1^2 (\gamma_1 m + 1)c^2 + 162(3\gamma_1 m + 1)^3 a^2)b^2 \right. \\ &\quad \left. + (162 - 2187(\gamma_1^3 m^3 + \gamma_1^2 m^2) + 1458\gamma_1 m)bc^3 + 81(3\gamma_1 m + 1)^3 (c^2 - a^2)^2] d^2 + 4(3\gamma_1 m + 1)^3 (a^2 + b^2 - c^2) \right\}, \\ V(\xi) &= A_0 + B_1 \cot\left(\frac{\theta}{2}\right), \end{aligned} \quad (52)$$

in which a, b, c, α and β are free values. By utilizing the (49) and Group 1, one obtain

$$\begin{aligned} v_1(\xi) &= \mp \frac{d(-6\gamma_1 m i - 2i + 9ad + 27ad\gamma_1 m)}{18\gamma_1 \gamma_2 d(-2\gamma_1 \gamma_2)^{-(1/2)}} \pm \frac{d}{\sqrt{-2\gamma_1 \gamma_2}} \left[\frac{a}{b-c} - \frac{\sqrt{c^2 - b^2 - a^2}}{b-c} \tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}(\xi + C)\right) \right]^{-1}, \\ U_1(\xi) &= \exp\left(\frac{i}{6561bc^2(b+c)\gamma_1 d^2} \left\{ [81(3\gamma_1 m + 1)^3 b^4 + (-6561m^2 \gamma_1^2 (\gamma_1 m + 1)c^2 + 162(3\gamma_1 m + 1)^3 a^2)b^2 \right. \right. \\ &\quad \left. \left. + (162 - 2187(\gamma_1^3 m^3 + \gamma_1^2 m^2) + 1458\gamma_1 m)bc^3 + 81(3\gamma_1 m + 1)^3 (c^2 - a^2)^2] d^2 + 4(3\gamma_1 m + 1)^3 (a^2 + b^2 - c^2) \right\} x - imt \right) \\ &\quad \times \left(\mp \frac{d(-6\gamma_1 m i - 2i + 9ad + 27ad\gamma_1 m)}{18\gamma_1 \gamma_2 d(-2\gamma_1 \gamma_2)^{-(1/2)}} \pm \frac{d}{\sqrt{-2\gamma_1 \gamma_2}} \left[\frac{a}{b-c} - \frac{\sqrt{c^2 - b^2 - a^2}}{b-c} \tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}(\xi + C)\right) \right]^{-1} \right). \end{aligned} \quad (53)$$

By utilizing equation (49) and Family 2, one get

$$\begin{aligned} V_2(\xi) &= \mp \frac{d(-6\gamma_1 m i - 2i + 9ad + 27ad\gamma_1 m)}{18\gamma_1 \gamma_2 d(-2\gamma_1 \gamma_2)^{-(1/2)}} \pm \frac{d}{\sqrt{-2\gamma_1 \gamma_2}} \left[\frac{a}{b-c} + \frac{\sqrt{a^2 + b^2 - c^2}}{b-c} \tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}(\xi + C)\right) \right]^{-1}, \\ U_2(\xi) &= \exp\left(\frac{i}{6561bc^2(b+c)\gamma_1 d^2} \left\{ [81(3\gamma_1 m + 1)^3 b^4 + (-6561m^2 \gamma_1^2 (\gamma_1 m + 1)c^2 + 162(3\gamma_1 m + 1)^3 a^2)b^2 \right. \right. \\ &\quad \left. \left. + (162 - 2187(\gamma_1^3 m^3 + \gamma_1^2 m^2) + 1458\gamma_1 m)bc^3 + 81(3\gamma_1 m + 1)^3 (c^2 - a^2)^2] d^2 + 4(3\gamma_1 m + 1)^3 (a^2 + b^2 - c^2) \right\} x - imt \right) \\ &\quad \times \left(\mp \frac{d(-6\gamma_1 m i - 2i + 9ad + 27ad\gamma_1 m)}{18\gamma_1 \gamma_2 d(-2\gamma_1 \gamma_2)^{-(1/2)}} \pm \frac{d}{\sqrt{-2\gamma_1 \gamma_2}} \left[\frac{a}{b-c} + \frac{\sqrt{a^2 + b^2 - c^2}}{b-c} \tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}(\xi + C)\right) \right]^{-1} \right). \end{aligned} \quad (54)$$

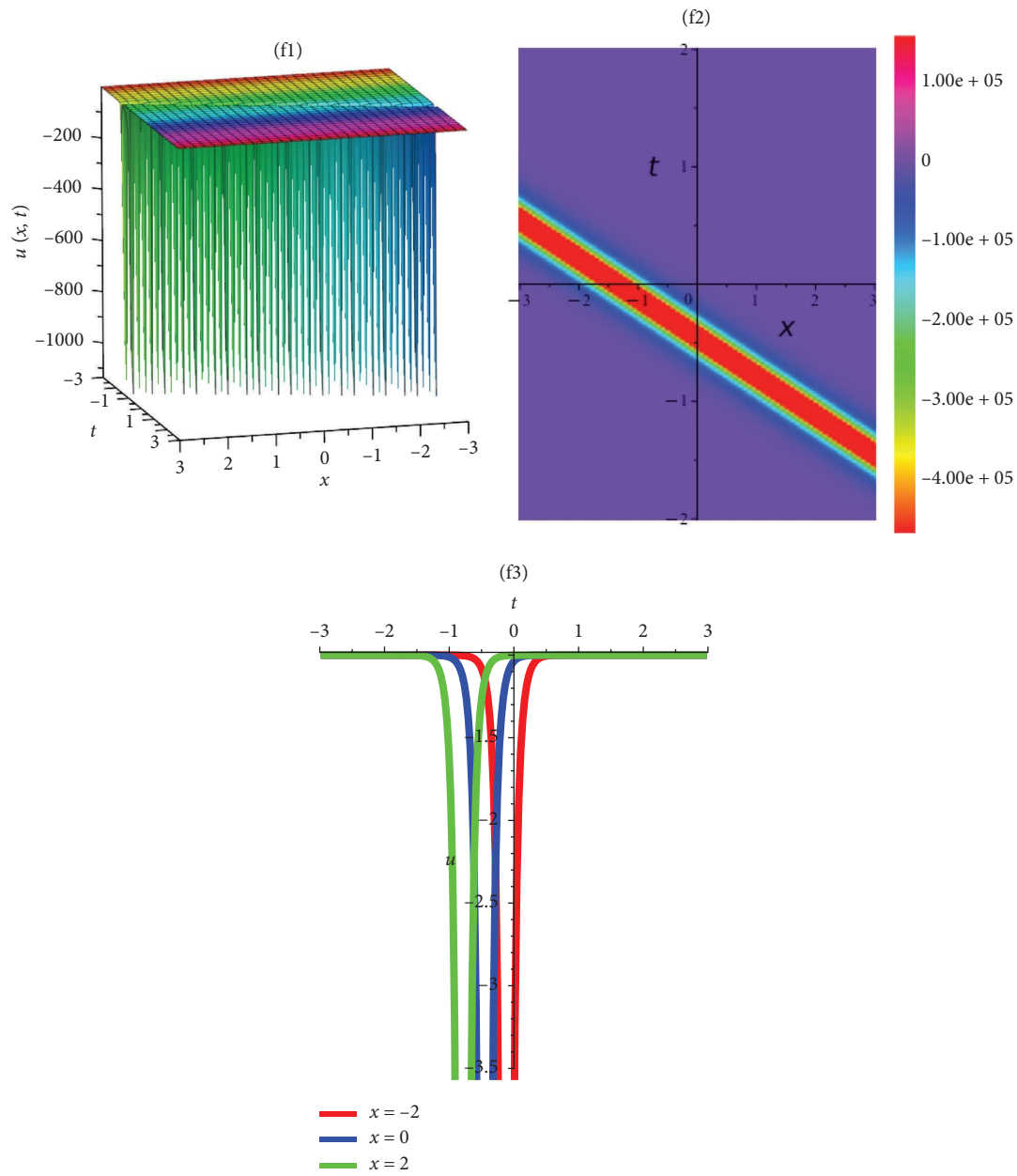


FIGURE 4: Graph of kink wave solution $v_1(x, y, t)$ (46) for $A = 1.5, B = -1, A_0 = 1, A_1 = 2, c = 4, \eta = 3, \mu = -289/8192$ and $y = 1$.

The periodic wave has one ascending mode in the (x, t) -plane in Figure 5 which is represented by a 3D plot, density plot, and 2D plot. Also, from Figure 6 one can discover

that the soliton solution has made in the (x, t) -plane, which is represented by a 3D plot, density plot, and 2D plot.

Via utilizing equation (49) and Family 5, one give

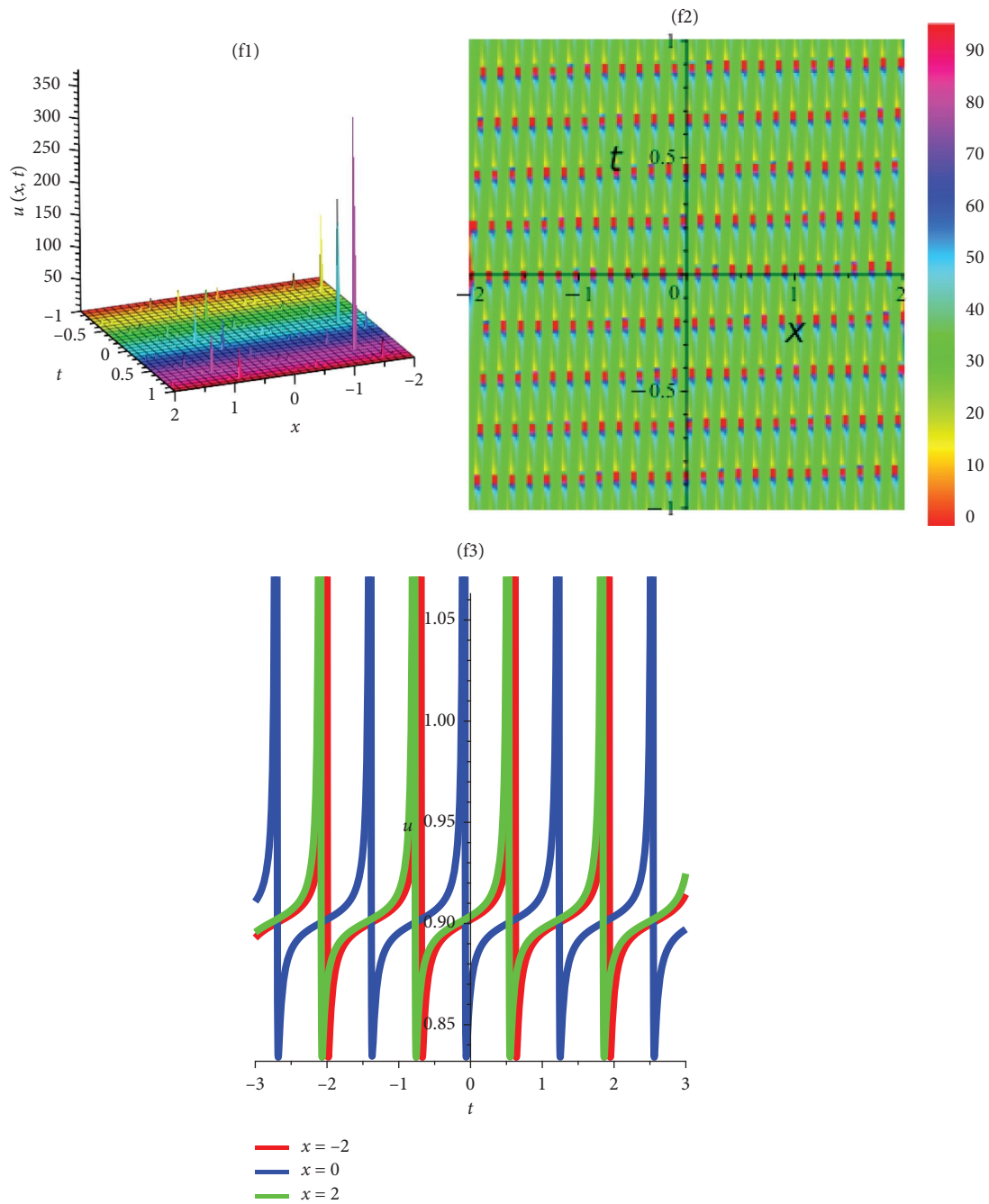


FIGURE 5: Graph of periodic solution $|u_1(x, t)|$ (53) for $\omega = -1, a = 1, b = 1, c = 5, \gamma_1 = 2, \gamma_2 = 3$ and $m = 2$.

$$\begin{aligned}
 V_3(\xi) &= \mp \frac{d(-6\gamma_1 m i - 2i)}{18\gamma_1 \gamma_2 d(-2\gamma_1 \gamma_2)^{-(1/2)}} \pm \frac{d}{\sqrt{-2\gamma_1 \gamma_2}} \frac{\sqrt{b^2 - c^2}}{b - c} \coth\left(\frac{\sqrt{b^2 - c^2}}{2}(\xi + C)\right), \\
 U_3(\xi) &= \exp\left(\frac{i}{6561bc^2(b+c)\gamma_1 d^2} \left\{ [81(3\gamma_1 m + 1)^3 b^4 + (-6561m^2 \gamma_1^2 (\gamma_1 m + 1)c^2)b^2 \right. \right. \\
 &\quad \left. \left. + (162 - 2187(\gamma_1^3 m^3 + \gamma_1^2 m^2) + 1458\gamma_1 m)bc^3 + 81(3\gamma_1 m + 1)^3 c^4] d^2 + 4(3\gamma_1 m + 1)^3(b^2 - c^2) \right\} x - imt \right) \\
 &\quad \times \left(\mp \frac{d(-6\gamma_1 m i - 2i)}{18\gamma_1 \gamma_2 d(-2\gamma_1 \gamma_2)^{-(1/2)}} \pm \frac{d}{\sqrt{-2\gamma_1 \gamma_2}} \frac{\sqrt{b^2 - c^2}}{b - c} \coth\left(\frac{\sqrt{b^2 - c^2}}{2}(\xi + C)\right) \right).
 \end{aligned} \tag{55}$$

By utilizing equation (49) and Family 8, one get

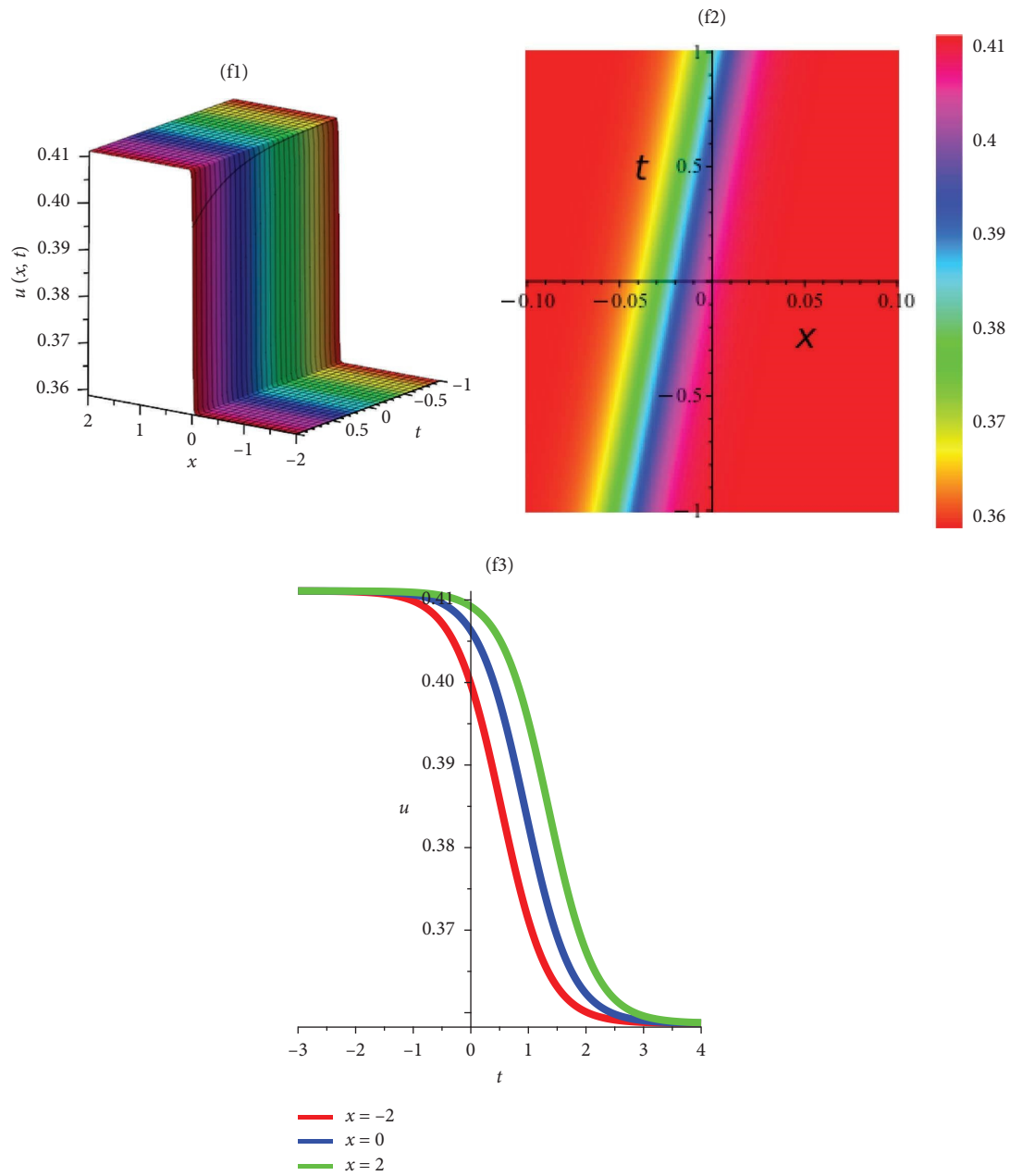


FIGURE 6: Graph of soliton solution $|u_2(x, t)|$ (54) for $\omega = -1, a = 3, b = 1, c = 2, \gamma_1 = 2, \gamma_2 = 3$ and $m = 2$.

$$\begin{aligned}
 V_4(\xi) &= \mp \frac{d(-6\gamma_1 m i - 2i + 9ad + 27ad\gamma_1 m)}{18\gamma_1 \gamma_2 d(-2\gamma_1 \gamma_2)^{-(1/2)}} \pm \frac{d}{\sqrt{-2\gamma_1 \gamma_2}} \frac{a^2(\xi + C)}{(b + c)(a(\xi + C) + 2)}, \\
 u_4(\xi) &= \exp\left(\frac{i}{6561bc^2(b + c)\gamma_1 d^2} \left\{ [81(3\gamma_1 m + 1)^3 b^4 + (-6561m^2 \gamma_1^2 (\gamma_1 m + 1)c^2 + 162(3\gamma_1 m + 1)^3 a^2)b^2 \right. \right. \\
 &\quad \left. \left. + (162 - 2187(\gamma_1^3 m^3 + \gamma_1^2 m^2) + 1458\gamma_1 m)bc^3 + 81(3\gamma_1 m + 1)^3(c^2 - a^2)^2 \right\} d^2 \right\} x - imt) \\
 &\quad \times \left(\mp \frac{d(-6\gamma_1 m i - 2i + 9ad + 27ad\gamma_1 m)}{18\gamma_1 \gamma_2 d(-2\gamma_1 \gamma_2)^{-(1/2)}} \pm \frac{d}{\sqrt{-2\gamma_1 \gamma_2}} \frac{a^2(\xi + C)}{(b + c)(a(\xi + C) + 2)} \right),
 \end{aligned} \tag{56}$$

where

$$\begin{aligned} \xi = & \left[\frac{(3\gamma_1 m + 1)^3 d a^4}{27 b c^2 (b + c) \gamma_1^2} + \frac{1}{2187 b c^2 (b + c) \gamma_1^2 d} \left\{ \left[(3\gamma_1 m + 1)^3 a^2 [162(b^2 - c^2) + 4] \right] + (81(3\gamma_1 m + 1)^3 d^2 b^4 \right. \right. \\ & + (-243(3\gamma_1 m + 1)(9\gamma_1^2 m^2 + 6\gamma_1 m - 2)d^2 c^2 + 4(3\gamma_1 m + 1)^3) b^2 \\ & - 81(3\gamma_1 m + 4)(3\gamma_1 m - 2)(3\gamma_1 m + 1)d^2 c^3 b + 81(3\gamma_1 m + 1)^3 d^2 c^4 - 4(3\gamma_1 m + 1)^3 c^2 \left. \right\} \right] x \\ & \pm \left[\frac{1}{9 b c (b + c)} \sqrt{b(b + c) \left[\mp 3(a^2 + b^2 - c^2) + \sqrt{9(a^2 + b^2)^2 + (9c^4 - 18a^2 c^2 - 42b^2 c^2 - 24bc^3)} \right]} \right] t. \end{aligned} \quad (57)$$

3.2.2. Set II

$$\begin{aligned} \omega = \omega, A_0 = & \pm \frac{6m\gamma_1 i + 2i + 9\omega\gamma_1 a}{9\sqrt{-2\gamma_1\gamma_2}}, A_1 = \pm \frac{\sqrt{-2\gamma_1\gamma_2}}{2\gamma_2} (b - c)\omega, B_1 = 0, m = m, \\ \alpha = & -\frac{1}{54\gamma_1 (b - c)} \left\{ 27a^2 \omega^3 \gamma_1^2 (b - c) + (27\omega^3 \gamma_1^2 b^2 (b - c) \right. \\ & + (81\omega^2 \gamma_1^2 c^2 - 16 + 18m^2 \gamma_1^2 + 12m\gamma_1) b \omega 27\omega^3 \gamma_1^2 c^3 + (16 - 18m^2 \gamma_1^2 - 12m\gamma_1) \omega c \left. \right\}, \\ k = & -\frac{1}{162(b - c)\gamma_1} \left\{ [a^2(b - c) + b^3 - b^2 c - bc^2 + c^3] (81m\gamma_1^3 \omega^2 + 27\gamma_1^2 \omega^2) \right. \\ & + (B - c) (54m^3 \gamma_1^3 + 54m^2 \gamma_1^2 - 36, \gamma_1 - 4) \left. \right\} - \frac{2bc^2 \omega^3 \gamma_1 a i}{b - c}, \quad v(\xi) = A_0 + A_1 \tan\left(\frac{\Phi(\xi)}{2}\right), \end{aligned} \quad (58)$$

in which a, b, c, α and β are free values. By utilizing equation (58) and Group 1, one become

$$\begin{aligned} v_5(\xi) = & \pm \frac{6m\gamma_1 i + 2i + 9\omega\gamma_1 a}{9\sqrt{-2\gamma_1\gamma_2}} \pm \frac{\sqrt{-2\gamma_1\gamma_2}}{2\gamma_2} \omega \left[a - \sqrt{c^2 - b^2 - a^2} \tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2} (\xi + C)\right) \right], \\ U_5(\xi) = & \exp\left(\left[-\frac{i}{162(b - c)\gamma_1} \left\{ [a^2(b - c) + b^3 - b^2 c - bc^2 + c^3] (81m\gamma_1^3 \omega^2 + 27\gamma_1^2 \omega^2) \right. \right. \right. \\ & + (B - c) (54m^3 \gamma_1^3 + 54m^2 \gamma_1^2 - 36, \gamma_1 - 4) \left. \right\} - \frac{2bc^2 \omega^3 \gamma_1 a i}{b - c} \left. \right] x - imt \Bigg) \\ & \times \left(\pm \frac{6m\gamma_1 i + 2i + 9\omega\gamma_1 a}{9\sqrt{-2\gamma_1\gamma_2}} \pm \frac{\sqrt{-2\gamma_1\gamma_2}}{2\gamma_2} \omega \left[a - \sqrt{c^2 - b^2 - a^2} \tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2} (\xi + C)\right) \right] \right). \end{aligned} \quad (59)$$

Via utilizing equation (58) and Family 2, one give

$$\begin{aligned}
 V_6(\xi) &= \pm \frac{6m\gamma_1 i + 2i + 9\omega\gamma_1 a}{9\sqrt{-2\gamma_1\gamma_2}} \pm \frac{\sqrt{-2\gamma_1\gamma_2}}{2\gamma_2} \omega \left[a + \sqrt{a^2 + b^2 - c^2} \tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2} (\xi + C)\right) \right], \\
 U_6(\xi) &= \exp\left(\left[-\frac{i}{162(b-c)\gamma_1} \{[a^2(b-c) + b^3 - b^2c - bc^2 + c^3](81m\gamma_1^3\omega^2 + 27\gamma_1^2\omega^2)\right.\right. \\
 &\quad \left.\left.+ (B-c)(54m^3\gamma_1^3 + 54m^2\gamma_1^2 - 36\gamma_1 - 4)\right\} - \frac{2bc^2\omega^3\gamma_1 a i}{b-c}\right] x - imt) \\
 &\quad \times \left(\pm \frac{6m\gamma_1 i + 2i + 9\omega\gamma_1 a}{9\sqrt{-2\gamma_1\gamma_2}} \pm \frac{\sqrt{-2\gamma_1\gamma_2}}{2\gamma_2} \omega \left[a + \sqrt{a^2 + b^2 - c^2} \tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2} (\xi + C)\right) \right]\right).
 \end{aligned} \tag{60}$$

By utilizing equation (58) and Family 5, one get

$$\begin{aligned}
 V_7(\xi) &= \pm \frac{6m\gamma_1 i + 2i}{9\sqrt{-2\gamma_1\gamma_2}} \pm \frac{\sqrt{-2\gamma_1\gamma_2}}{2\gamma_2} \omega \sqrt{b^2 - c^2} \tanh\left(\frac{\sqrt{b^2 - c^2}}{2} (\xi + C)\right), \\
 U_7(\xi) &= \exp\left(\left[-\frac{i}{162(b-c)\gamma_1} \{[b^3 - b^2c - bc^2 + c^3](81m\gamma_1^3\omega^2 + 27\gamma_1^2\omega^2) + (B-c)(54m^3\gamma_1^3 + 54m^2\gamma_1^2 - 36\gamma_1 - 4)\right\}\right. \\
 &\quad \left.\right] x - imt) \\
 &\quad \times \left(\pm \frac{6m\gamma_1 i + 2i}{9\sqrt{-2\gamma_1\gamma_2}} \pm \frac{\sqrt{-2\gamma_1\gamma_2}}{2\gamma_2} \omega \sqrt{b^2 - c^2} \tanh\left(\frac{\sqrt{b^2 - c^2}}{2} (\xi + C)\right)\right).
 \end{aligned} \tag{61}$$

By utilizing equation (58) and Family 8, one get

$$\begin{aligned}
 V_8(\xi) &= \pm \frac{6m\gamma_1 i + 2i + 9\omega\gamma_1 a}{9\sqrt{-2\gamma_1\gamma_2}} \pm \frac{\sqrt{-2\gamma_1\gamma_2}}{2\gamma_2} \omega \frac{(b^2 - c^2)(a(\xi + C) + 2)}{a^2(\xi + C)}, \\
 U_8(\xi) &= \exp\left(\frac{i}{6561bc^2(b+c)\gamma_1 d^2} \{[81(3\gamma_1 m + 1)^3 b^4 + (-6561m^2\gamma_1^2(\gamma_1 m + 1)c^2 + 162(3\gamma_1 m + 1)^3 a^2)b^2\right. \\
 &\quad \left.+ (162 - 2187(\gamma_1^3 m^3 + \gamma_1^2 m^2) + 1458\gamma_1 m)bc^3 + 81(3\gamma_1 m + 1)^3(c^2 - a^2)^2\right] d^2\} x - imt) \\
 &\quad \times \left(\pm \frac{6m\gamma_1 i + 2i + 9\omega\gamma_1 a}{9\sqrt{-2\gamma_1\gamma_2}} \pm \frac{\sqrt{-2\gamma_1\gamma_2}}{2\gamma_2} \omega \frac{(b^2 - c^2)(a(\xi + C) + 2)}{a^2(\xi + C)}\right),
 \end{aligned} \tag{62}$$

where

$$\xi = \left[-\frac{1}{54\gamma_1(b-c)} \{27a^2\omega^3\gamma_1^2(b-c) + (27\omega^3\gamma_1^2b^2(b-c) + (81\omega^2\gamma_1^2c^2 - 16 + 18m^2\gamma_1^2 + 12m\gamma_1)b\omega 27\omega^3\gamma_1^2c^3 + (16 - 18m^2\gamma_1^2 - 12m\gamma_1)\omega c)\} \right] x + \omega t. \tag{63}$$

4. The $\exp(-\Omega(\eta))$ -Expansion Method

Handling the investigated model through the $\exp(-\Omega(\eta))$ -expansion method gets the following steps as mentioned earlier as follows:

Step 1.

$$\mathcal{S}_1(\mathfrak{R}, \mathfrak{R}_x, \mathfrak{R}_t, \mathfrak{R}_{xx}, \mathfrak{R}_{tt}, \dots) = 0. \quad (64)$$

Where $\mathcal{S}$ is a polynomial of $\mathfrak{R}$ and its partial derivatives.

Step 2. Firstly, by utilizing traveling wave transformation

$$\xi = x + y + \eta t. \quad (65)$$

Where η is the non-zero arbitrary value, allows to diminish equation (64) to an ODE of $\mathfrak{R} = \mathfrak{R}(\xi)$ in the below form

$$\mathcal{S}_2(\mathfrak{R}, \theta_1 \mathfrak{R}', \theta_2 \mathfrak{R}', \theta_1^2 \mathfrak{R}'', \theta_2^2 \mathfrak{R}'', \dots) = 0, \quad (66)$$

Step 3. The generated solutions of (3) are:

$$\mathfrak{R}(\xi) = \frac{\sum_{i=0}^N A_i (e^{-\phi(\xi)})^i}{\sum_{i=0}^M B_i (e^{-\phi(\xi)})^i}. \quad (67)$$

Where A_j, B_j ($0 \leq j \leq N, M$), are the parameters to be determined $A_N, B_M \neq 0$, and, $\Phi = \Phi(\xi)$ satisfying the ODE given below

$$\phi' = p_1 e^{\phi(\xi)} + e^{-\phi(\xi)} + p_2, \quad (68)$$

The particular solutions of equation (68) will be read as

Solution-1: When $w_1 \neq 0$ and $p_2^2 - 4p_1 > 0$, therefore we attain

$$\phi(\eta) = \ln(-(\sqrt{p_2^2 - 4p_1} / (2p_1)) \tanh(\sqrt{p_2^2 - 4p_1} / 2 (\xi + E)) - p_2 / (2p_1)).$$

Solution-2: When $p_1 \neq 0$ and $p_2^2 - 4p_1 < 0$, therefore we attain

$$\phi(\eta) = \ln(\sqrt{-p_2^2 + 4p_1} / (2p_1) \tan(\sqrt{-p_2^2 + 4p_1} / 2 (\xi + E)) - p_2 / (2p_1)).$$

Solution-3: When $p_1 = 0$, $p_2 \neq 0$, and $p_2^2 - 4p_1 > 0$, therefore we attain $\Phi(\eta) = -\ln(p_2 / (\exp(p_2(\xi + E)) - 1))$.

Solution-4: When $p_1 \neq 0$, $p_2 \neq 0$, and $p_2^2 - 4p_1 = 0$, therefore we attain $\phi(\eta) = \ln(-(2p_2(\xi + E) + 4) / (p_2^2(\xi + E)))$.

Solution-5: When $p_1 = 0$, $p_2 = 0$, and $p_2^2 - 4p_1 = 0$, therefore we attain $\phi(\eta) = \ln(\xi + E)$, where A_j ($0 \leq j \leq m$), E , p_2 and p_1 are also the constants to be explored later.

Step 4. By balancing the nonlinear ODE can obtain the values M and N .

Step 5. By solving the algebraic equations an get to the mentioned values in the above.

5. Application EFM

Case 1. For first example, the set of category of solutions with $N = 1, M = 0$: with inserting

$$\mathfrak{R}(\xi) = \frac{\sum_{i=0}^1 A_i (e^{-\phi(\xi)})^i}{\sum_{i=0}^0 B_i (e^{-\phi(\xi)})^i}, \quad (69)$$

into (16) we have the nonlinear algebra system as

$$\begin{aligned} -4\mu A_1 B_0^2 + A_1^3 &= 0, \\ -2\mu A_1 B_0^2 p_1 p_2 + \eta A_1 B_0^2 p_1 + A A_0 B_0^2 + A_0^3 &= 0, \\ -6\mu A_1 B_0^2 p_2 + \eta A_1 B_0^2 + 3 A_0 A_1^2 &= 0, \\ -2\mu A_1 B_0^2 p_2^2 + \eta A_1 B_0^2 p_2 - 4\mu A_1 B_0^2 p_1 + A A_1 B_0^2 + 3 A_0^2 A_1 &= 0. \end{aligned} \quad (70)$$

Then, the results will be reached as.

5.1. Set I

$$\begin{aligned} \eta &= 3\sqrt{-A\mu}, A_0 = -\frac{1}{2} \frac{B_0(-2\mu p_2 + \sqrt{-A\mu})}{\sqrt{\mu}}, A_1 = 2\sqrt{\mu} B_0, B_0 = B_0, \\ p_1 &= \frac{1}{16} \frac{4\mu p_2^2 + A}{\mu}, p_2 = p_2. \end{aligned} \quad (71)$$

As a result by (Groups 1-4), the soliton solutions is given by

$$\mathfrak{R}_1(x, y, t) = -\frac{1}{2} \frac{(-2\mu p_2 + \sqrt{-A\mu})}{\sqrt{\mu}} 2\sqrt{\mu} \frac{1}{\left(-\left(\sqrt{p_2^2 - 4p_1}/(2p_1)\right) \tanh\left(\left(\sqrt{p_2^2 - 4p_1}/2\right)(\xi + E)\right) - ((p_2)/(2p_1))\right)}, \quad (72)$$

$$\mathfrak{R}_2(x, y, t) = -\frac{1}{2} \frac{(-2\mu p_2 + \sqrt{-A\mu})}{\sqrt{\mu}} 2\sqrt{\mu} \frac{1}{\left(\left(\sqrt{-p_2^2 + 4p_1}/(2p_1)\right) \tanh\left(\left(\sqrt{-p_2^2 + 4p_1}/2\right)(\xi + E)\right) - ((p_2)/(2p_1))\right)}, \quad (73)$$

$$\mathfrak{R}_3(x, y, t) = -\frac{1}{2} \frac{(-2\mu p_2 + \sqrt{-A\mu})}{\sqrt{\mu}} + 2\sqrt{\mu} \left(\frac{p_2}{\exp(p_2(\xi + E)) - 1} \right), \quad (74)$$

$$\mathfrak{R}_4(x, y, t) = -\frac{1}{2} \frac{(-2\mu p_2 + \sqrt{-A\mu})}{\sqrt{\mu}} + 2\sqrt{\mu} \frac{1}{\left(-((2p_2(\xi + E) + 4)/(p_2^2(\xi + E)))\right)}. \quad (75)$$

Case 2. For the second example, the set of category of solutions with $N = 1, M = 0$: with inserting into (16) we have the nonlinear algebra system as

$$\mathfrak{R}(\xi) = \frac{\sum_{i=0}^1 A_i (e^{-\phi(\xi)})^i}{\sum_{i=0}^0 B_i (e^{-\phi(\xi)})^i}, \quad (76)$$

$$\begin{aligned} &18i\gamma_1^2\omega^3A_1B_0^2 + 9i\omega A_1^3\gamma_1\gamma_2 = 0, \\ &9m\omega^2A_1B_0^2\gamma_1^2p_1p_2 + 3i\omega^3A_1B_0^2\gamma_1^2p_1p_2^2 - 6im\omega A_1B_0^2\gamma_1p_1 + 9i\omega A_0^2A_1\gamma_1\gamma_2p_1 + A_0^3\gamma_2 \\ &\quad - 3m^3A_0B_0^2\gamma_1^2 - 3m^2A_0B_0^2\gamma_1 - 3kA_0B_0^2\gamma_1 + 3mA_0^3\gamma_1\gamma_2 + 6i\omega^3A_1B_0^2\gamma_1^2p_1^2 - 9im^2A_1B_0^2\gamma_1^2p_1 \\ &\quad - 3i\alpha A_1B_0^2\gamma_1p_1 + 3\omega^2A_1B_0^2\gamma_1p_1p_2 = 0, \\ &A_1^3\gamma_2 + 3mA_1^3\gamma_1\gamma_2 + 9i\omega A_1^3\gamma_1\gamma_2p_2 + 18i\omega A_0A_1^2\gamma_1\gamma_2 + 36i\omega^3A_1B_0^2\gamma_1^2p_2 + 6\omega^2A_1B_0^2\gamma_1 + 18m\omega^2A_1B_0^2\gamma_1^2 = 0, \\ &3A_0A_1^2\gamma_2 + 18i\omega A_0A_1^2\gamma_1\gamma_2p_2 - 9im^2A_1B_0^2\gamma_1^2 - 3i\alpha A_1B_0^2\gamma_1 + 9i\omega A_1^3\gamma_1\gamma_2p_1 \\ &\quad - 6im\omega A_1B_0^2\gamma_1 + 9i\omega A_0^2A_1\gamma_1\gamma_2 + 21i\omega^3A_1B_0^2\gamma_1^2p_2^2 + 24i\omega^3A_1B_0^2\gamma_1^2p_1 \\ &\quad + 27m\omega^2A_1B_0^2\gamma_1^2p_2 + 9mA_0A_1^2\gamma_1\gamma_2 + 9\omega^2A_1B_0^2\gamma_1p_2 = 0, \\ &3A_0^2A_1\gamma_2 + 3i\omega^3A_1B_0^2\gamma_1^2p_2^3 - 6im\omega A_1B_0^2\gamma_1p_2 - 9im^2A_1B_0^2\gamma_1^2p_2 + 9i\omega A_0^2A_1\gamma_1\gamma_2p_2 \\ &\quad - 3m^3A_1B_0^2\gamma_1^2 - 3m^2A_1B_0^2\gamma_1 - 3kA_1B_0^2\gamma_1 + 24i\omega^3A_1B_0^2\gamma_1^2p_1p_2 - 3i\alpha A_1B_0^2\gamma_1p_2 \\ &\quad + 18i\omega A_0A_1^2\gamma_1\gamma_2p_1 + 9m\omega^2A_1B_0^2\gamma_1^2p_2^2 + 18m\omega^2A_1B_0^2\gamma_1^2p_1 + 9mA_0^2A_1\gamma_1\gamma_2 \\ &\quad + 3\omega^2A_1B_0^2\gamma_1p_2^2 + 6\omega^2A_1B_0^2\gamma_1p_1 = 0. \end{aligned} \quad (77)$$

Then, the results will be reached as.

5.2. Set I

$$\begin{aligned} \alpha &= \frac{234m^2\omega\gamma_1^2 - 243m^2\gamma_1^2 - 6m\omega\gamma_1 + 26\omega}{81\gamma_1}, k = -\frac{27m^3\gamma_1^3 + 27m^2\gamma_1^2 - 72m\gamma_1 - 8}{243\gamma_1^2}, \\ m &= m, \omega = \omega, A_0 = \frac{1}{9} \frac{B_0(6im\gamma_1 - 9\omega\gamma_1p_2 + 2i)}{\gamma_2} \frac{1}{\sqrt{-2((\gamma_1)/(\gamma_2))}}, A_1 = \sqrt{-2\frac{\gamma_1}{\gamma_2}} B_0\omega, \\ p_1 &= \frac{81\omega^2\gamma_1^2p_2^2 + 36m^2\gamma_1^2 + 24m\gamma_1 + 4}{324\omega^2\gamma_1^2}, B_0 = B_0, p_2 = p_2. \end{aligned} \quad (78)$$

As a result by (Groups 1–4), the soliton solutions is given by

$$\mathfrak{R}_1(x, t) = e^{i(-((27m^3\gamma_1^3 + 27m^2\gamma_1^2 - 72m\gamma_1 - 8)/(243\gamma_1^2))x + mt)} \left[\frac{1}{9} \frac{(6im\gamma_1 - 9\omega\gamma_1 p_2 + 2i)}{\gamma_2} \frac{1}{\sqrt{-2((\gamma_1)/(\gamma_2))}} \right. \\ \left. + \sqrt{-2} \frac{\gamma_1}{\gamma_2} \omega \frac{1}{\left(-\left(\sqrt{p_2^2 - 4p_1}/(2p_1)\right) \tanh\left(\left(\sqrt{p_2^2 - 4p_1}/2\right)(\xi + E)\right) - ((p_2)/(2p_1))\right)} \right], \quad (79)$$

$$\mathfrak{R}_2(x, t) = e^{i(-((27m^3\gamma_1^3 + 27m^2\gamma_1^2 - 72m\gamma_1 - 8)/(243\gamma_1^2))x + mt)} \left[\frac{1}{9} \frac{(6im\gamma_1 - 9\omega\gamma_1 p_2 + 2i)}{\gamma_2} \frac{1}{\sqrt{-2((\gamma_1)/(\gamma_2))}} \right. \\ \left. + \sqrt{-2} \frac{\gamma_1}{\gamma_2} \omega \frac{1}{\left(\left(\sqrt{-p_2^2 + 4p_1}/(2p_1)\right) \tanh\left(\left(\sqrt{-p_2^2 + 4p_1}/2\right)(\xi + E)\right) - ((p_2)/(2p_1))\right)} \right], \quad (80)$$

$$\mathfrak{R}_3(x, t) = e^{i(-((27m^3\gamma_1^3 + 27m^2\gamma_1^2 - 72m\gamma_1 - 8)/(243\gamma_1^2))x + mt)} \left[\frac{1}{9} \frac{(6im\gamma_1 - 9\omega\gamma_1 p_2 + 2i)}{\gamma_2} \frac{1}{\sqrt{-2((\gamma_1)/(\gamma_2))}} \right. \\ \left. + \sqrt{-2} \frac{\gamma_1}{\gamma_2} \omega \left(\frac{p_2}{\exp(p_2(\xi + E)) - 1} \right) \right], \quad (81)$$

$$\mathfrak{R}_4(x, t) = e^{i(-((27m^3\gamma_1^3 + 27m^2\gamma_1^2 - 72m\gamma_1 - 8)/(243\gamma_1^2))x + mt)} \left[\frac{1}{9} \frac{(6im\gamma_1 - 9\omega\gamma_1 p_2 + 2i)}{\gamma_2} \frac{1}{\sqrt{-2((\gamma_1)/(\gamma_2))}} \right. \\ \left. + \sqrt{-2} \frac{\gamma_1}{\gamma_2} \omega \frac{1}{\left(-\left((2p_2(\xi + E) + 4)/(p_2^2(\xi + E))\right)\right)} \right], \quad (82)$$

where $\xi = (234m^2\omega\gamma_1^2 - 243m^2\gamma_1^2 - 6m\omega\gamma_1 + 26\omega)/(81\gamma_1)x + \omega t$. 5.3. Set II

$$\alpha = \frac{1}{6} \frac{-3\omega^3\gamma_1^2 p_2^2 + 12\omega^3\gamma_1^2 p_1 + 4\omega - 2}{\gamma_1}, k = -\frac{2}{27\gamma_1^2}, \\ m = -\frac{1}{3}\gamma_1^{-1}, \omega = \omega, A_0 = -\frac{\omega B_0\gamma_1 p_2}{\gamma_2} \left(\sqrt{-2} \frac{\gamma_1}{\gamma_2} \right)^{-1}, A_1 = \sqrt{-2} \frac{\gamma_1}{\gamma_2} B_0 \omega, \\ p_1 = p_1, B_0 = B_0, p_2 = p_2. \quad (83)$$

As a result by (Groups 1–4), the soliton solutions is given by

$$\mathfrak{R}_1(x, t) = e^{i(-2/(27\gamma_1^2))x - (1/3)\gamma_1^{-1}t} \left[\frac{\omega\gamma_1 p_2}{\gamma_2} \left(\sqrt{-2\frac{\gamma_1}{\gamma_2}} \right)^{-1} + \sqrt{-2\frac{\gamma_1}{\gamma_2}} \omega \frac{1}{\left(-\left(\sqrt{p_2^2 - 4p_1}/(2p_1) \right) \tanh\left(\left(\sqrt{p_2^2 - 4p_1}/2 \right) (\xi + E) \right) - ((p_2)/(2p_1)) \right)} \right], \quad (84)$$

$$\mathfrak{R}_2(x, t) = e^{i(-2/(27\gamma_1^2))x - (1/3)\gamma_1^{-1}t} \left[\frac{\omega\gamma_1 p_2}{\gamma_2} \left(\sqrt{-2\frac{\gamma_1}{\gamma_2}} \right)^{-1} + \sqrt{-2\frac{\gamma_1}{\gamma_2}} \omega \frac{1}{\left(\left(\sqrt{-p_2^2 + 4p_1}/(2p_1) \right) \tan\left(\left(\sqrt{-p_2^2 + 4p_1}/2 \right) (\xi + E) \right) - ((p_2)/(2p_1)) \right)} \right], \quad (85)$$

$$\mathfrak{R}_3(x, t) = e^{i(-2/(27\gamma_1^2))x - (1/3)\gamma_1^{-1}t} \left[\frac{\omega\gamma_1 p_2}{\gamma_2} \left(\sqrt{-2\frac{\gamma_1}{\gamma_2}} \right)^{-1} + \sqrt{-2\frac{\gamma_1}{\gamma_2}} \omega \left(\frac{p_2}{\exp(p_2(\xi + E)) - 1} \right) \right], \quad (86)$$

$$\mathfrak{R}_4(x, t) = e^{i(-2/(27\gamma_1^2))x - (1/3)\gamma_1^{-1}t} \left[\frac{\omega\gamma_1 p_2}{\gamma_2} \left(\sqrt{-2\frac{\gamma_1}{\gamma_2}} \right)^{-1} + \sqrt{-2\frac{\gamma_1}{\gamma_2}} \omega \frac{1}{\left(-((2p_2(\xi + E) + 4)/(p_2^2(\xi + E))) \right)} \right], \quad (87)$$

where

$$\xi = (1/6)((-3\omega^3\gamma_1^2 p_2^2 + 12\omega^3\gamma_1^2 p_1 + 4\omega - 2)/\gamma_1)x + \omega t.$$

Remark 1 (Graphical analysis of the results). In this section, we will explain and provide visual representations of the solutions (one-soliton, periodic, and singular soliton) obtained from the RD Brusselator system and the RKL equation. Graphical demonstrations of soliton solutions contribute significantly to the understanding of nonlinear processes. We depict the solutions of the considered equation in various two- and three-dimensional profiles. Emerging fields of study, including biophysics, information science, optical fibre communication, and fluid mechanics, were stimulated by the investigation of nonlinear wave-bounded state structures. As well as providing theoretical support for the analysis of experimental results and valuable mathematical physics information, the investigation of the aforementioned issues is crucial in certain scientific disciplines. Soliton, dark soliton, bright soliton, kink, breather, lump, and rogue wave are the four broad classifications that roughly describe nonlinear wave types. A kink soliton generally corresponds to a physical phenomenon in the original system. Kink solitons are a type of nonlinear wave that can be observed in various physical systems. They represent a localized shift or change in a variable, and their behavior is governed by the interplay between nonlinear interactions and dispersion in the system. Reflecting common nonlinear characteristics in nature, solitons possess a number of intriguing particle and wave properties. Periodicity, which results from periodic coherence between atoms, is a widely recognised and fundamental property of nonlinear wave propagation. A solution that exhibits periodic behaviour in both the time and space dimensions is precisely described by the breather. Localised high waves in both space and time are represented by the rogue wave. In

numerous integrable models, lump solutions are implemented; they are a form of rational, exact solutions that are localised spatially within a specific region. Under specific conditions, the aforementioned nonlinear wave structures are capable of undergoing transformations that generate a multitude of new wave structures. Equations (1) and (2) have been analyzed by the different perspective in the existing literature. The system (1) denoted by equation (16) represents a biochemical model. In this study, we examine the RD Brusselator model, which represents a chemical RD process using the $\tan(\phi/2)$ -expansion approach. In a study conducted by Khan et al. ([5]), the authors examined the use of Bernstein polynomials in obtaining numerical solutions for the nonlinear fractional-order Brusselator system. Additionally, Wazwaz ([7]) employed the Adomian decomposition approach to solve the RD Brusselator model. Chowdhury and colleagues ([6]) used the homotopy perturbation technique as a computational approach to address the RD Brusselator system. Jafari et al. ([8]) used the variational iteration approach to examine the approximate solutions of the aforementioned system.

6. Conclusion

The technique of expanding $\tan(\theta/2)$ and the approach of expanding $\exp(-\Omega(\eta))$ approach were used to derive analytical solutions for equations involving chemical and biological processes. It was observed that the obtained solutions for the RKL equation exhibit satisfactory performance, as reported in the study by [15]. The phase portraits, three-dimensional graphs, two-dimensional graphs, and density plot were generated using the Maple program by specifying preset parameters. In addition, the system under consideration was yielded solutions in the form of tan-expansion functions, kink solitary waves, and solitary

waves. Furthermore, this paper was presented newly derived findings in a formal manner. The precise solutions included three distinct sorts of solutions: hyperbolic function solutions, trigonometric function solutions, and rational solutions. Additionally, the findings were indicated that the suggested methodologies exhibit greater consistency, competence, and superiority in the analysis of diverse nonlinear equations. This article presented the derivation of precise solutions that include any arbitrary function. Based on the principle of arbitrariness inherent in the function, the solutions may be derived by the amalgamation of trigonometric, exponential, hyperbolic, and rational functions. Hence, the findings that were derived in the aforementioned reference [15] may be regarded as specific instances of the conclusions presented in this study.

Appendix A

PACS: 02.60.Lj, 02.70.Wz, 02.90.+p, 04.30.Nk.

Data Availability Statement

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

J.M. conceptualization, investigation; software; O.A.İ. resources; validation, writing–review reading. The authors have contributed equally to this work.

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