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Unveiling the nexus: Computational thinking and mathematical modelling in K-12 education- a teacher-centric exploration

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ABSTRACT

This study explores how Computational Thinking (CT) components overlap with the phases of mathematical modelling within the context of a Teacher Development Course (TDC). The course was designed, developed, implemented, and assessed to enhance teachers' cognitive actions in integrating CT with mathematical modelling. This research study was conducted with three mathematics teachers and one computer science teacher. Data were collected through CT component worksheets and video recordings, and analysed based on Borromeo-Ferri's (2006) modelling cycle and the study's CT framework. The study's findings indicate that modelling processes enhanced teachers' CT skills, while CT components made the modelling process more structured and reflective, revealing a reciprocal relationship between modelling and CT. The study proposes an original interdisciplinary framework linking teachers' cognitive actions to CT integration, offering both theoretical and practical contributions.

1. Introduction

Computational thinking (CT) originated in Papert's (1980) constructivist view of learners transforming abstract ideas into tangible understanding, which then gained prominence through Wing's (2006) articulation of CT as a fundamental mode of thought. Rather than being confined just to coding, CT represents a way of reasoning that emphasises abstraction, decomposition, algorithmic design, and creativity to approach complex problems systematically (Li et al., 2020; Lodi & Martini, 2021). In mathematics education, CT enhances learners' capacity to analyse, represent, and solve problems across both digital and unplugged contexts, including programming, tool-based exploration, and project- or problem-based learning (Chan et al., 2022; Vazquez-Uscanga et al., 2025; Wang et al., 2021). Through these experiences, learners engage within an iterative and cyclical process that intertwines mathematical and computational reasoning, strengthening analytical and critical thinking, improving problem-solving skills, and fostering confidence in addressing complex mathematical tasks (Perez, 2018; Wu et al., 2024; Ye et al., 2023).

In this context, the expanding role of CT in mathematics education intersects with mathematical modelling, a meaningful arena for problem solving (Kallia et al., 2021). Mathematical modelling engages learners through tackling real-life problems, thus allowing them to structure, abstract, and interpret problems better through cognitive action (Borromeo Ferri, 2018; English, 2003; Schukajlow et al., 2023). CT reinforces these processes through shared cognitive mechanisms such as decomposition, algorithmic design, data

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processing, and debugging. These mechanisms help learners break down complex problems into more manageable parts, construct systematic solutions, and refine them through testing (Lehmann, 2023; Shin et al., 2025; Zhang et al., 2024). This reciprocal relationship fosters an epistemological synergy between CT and mathematical cognition (Ang, 2021; Sanford & Naidu, 2017) while supporting learners' reasoning across multiple stages of the modelling cycle (Dinçer Aksoy et al., 2025). However, despite these conceptual overlaps, the integration of CT into modelling practices remains limited, pointing to the need for further research on how CT components can systematically strengthen both students' and teachers' modelling competencies (Hermans et al., 2024; Musaeus & Musaeus, 2024). Addressing this need requires the design of targeted Teacher Development Courses (TDCs) that embed CT within modelling-based learning environments.

However, despite its recognised importance, CT remains largely absent from most TDCs. Educators therefore require support to integrate CT components effectively into existing and newly developed teaching practices. High-quality professional development for teachers should address their specific needs in terms of student impact, professional growth, and contextual relevance (Albez et al., 2020; Mumcu et al., 2022). Today's teachers face numerous challenges such as aligning instructional approaches (Rich et al., 2020), assessing CT components (Cutumisu et al., 2019), and achieving interdisciplinary integration (Vieyra et al., 2024), while computer science (CS) unplugged approaches (Lv et al., 2023) and constructivist learning environments (Lodi & Martini, 2021; Papert, 1980) help to enhance teachers' contextualised understanding of CT. The interdisciplinary nature of CT becomes more functional when integrated with mathematical modelling (Mumcu et al., 2023a; Mumcu et al., 2023b), since it supports cognitive, social, and affective engagement (Papert, 1980).

The current study aimed to design, implement, and evaluate a TDC targeting the integration of CT components into mathematics teaching through the prism of mathematical modelling. In the study, teachers collaboratively tackled modelling problems through unplugged computer science activities and modelling tasks framed by the cognitive perspective (Borromeo Ferri, 2006) and interweaving emerging cognitive processes with CT components. Although CT has received growing attention in mathematics education, there is still limited understanding of how CT components align with and interact across the different stages of mathematical modelling in authentic professional development settings. Addressing this gap, the current study aimed to investigate how CT can be systematically embedded into modelling-based teacher education. The following question guided the current research:

How does the mathematical modelling process overlap epistemologically with computational thinking (CT) components?

2. Theoretical background

2.1. Mathematical modelling

In mathematical modelling, learners construct mathematical representations to address real-world problems, then generate solutions grounded in structured reasoning (Doruk & Umay, 2011). Beyond being a procedural tool, modelling offers a more sophisticated form of problem solving (Blum & Niss, 1989), whilst from a cognitive perspective it highlights cognitive and metacognitive actions such as planning, monitoring, and evaluating that shape mathematical reasoning (Kaiser & Sriraman, 2006; Schukajlow et al., 2023). The modelling phases, which typically include problem analysis, mathematisation, solution, and interpretation, serve to analyse the unfolding cognitive mechanisms throughout the process. Since modelling requires learners to build internal representations of problems and to articulate their reasoning across phases, the process naturally fosters learners' development of both cognitive and metacognitive actions (Schukajlow et al., 2023).

Modelling is thus conceived as a means to solve complex problems, tackle ill-structured tasks, and extends beyond routine procedural norms (Niss et al., 2007). The process unfolds through iterative cycles involving stages such as understanding, simplifying,

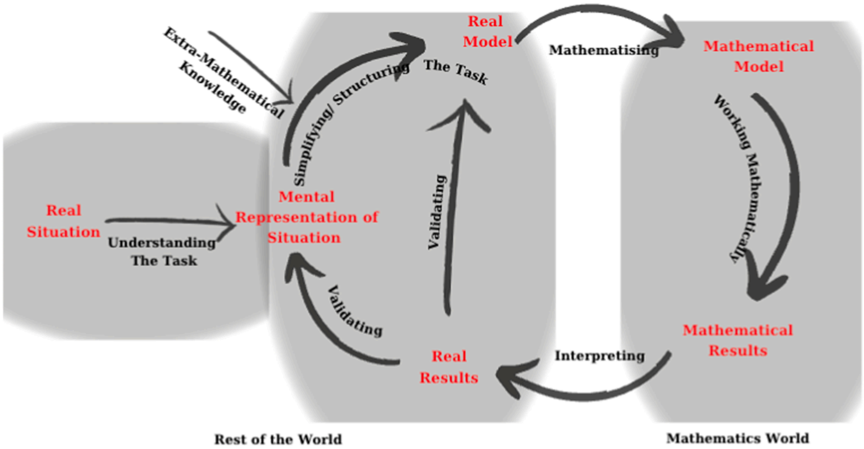

Fig. 1. Modelling cycle from a cognitive perspective (based on Borromeo Ferri, 2006).

mathematising, working mathematically, interpreting, and validating (Blum & Leiss, 2007; Galbraith & Stillman, 2006; Stillman, 2011). However, authentic problem solving rarely follows these idealised sequences, and students' pathways often reveal a more dynamic, non-linear route shaped by emergent reasoning (Borromeo Ferri, 2007). Working to build upon this view, the current study adopted Borromeo Ferri's (2006) framework in order to capture authentic cognitive processes and to examine the role of computational thinking within them (see Fig. 1).

Fig. 1 illustrates the cognitive phases of the modelling cycle adopted in the current study. Scholars have described these phases as idealised stages within a structured sequence, and with each serving a distinct cognitive function within the broader modelling process (Blum & Leiss, 2007; Borromeo Ferri, 2007; Niss et al., 2007). The cycle begins with the *understanding* phase, where learners construct a coherent situation model of the real-world context, identify givens, constraints, and aims, and prepare for subsequent modelling activities (Leiss et al., 2019). In the *simplifying* phase, complexity is reduced by making justified assumptions, identifying relevant variables, and specifying missing quantities in order to construct a real model (Geiger et al., 2021; Krawitz et al., 2018). The *mathematising* phase involves mapping the real model to mathematical structures by choosing appropriate representations, relations, and operations that lead to a workable mathematical model (Niss, 2010). In the *working mathematically* phase, learners operate within the mathematical model and make use of the relevant concepts, procedures, and tools necessary to produce the required mathematical results (Goos, 2002). The *interpreting* phase then requires the mathematical results to be translated back into the real-world context, assigning meaning, and determining an appropriate degree of accuracy (Galbraith et al., 2017). Finally, in the *validating* phase, learners check the plausibility of generated results against the assumptions and situation model, revisiting earlier phases if necessary, which reflects the iterative nature of the cycle (Blum & Borromeo Ferri, 2009; Czocher, 2018). Within this perspective, the modelling phases can serve as an analytical lens to examine how computational thinking components emerge throughout the process.

2.2. CT components and mathematics education

A systematic review examining the current status, outcomes, and implications of mathematics and CT integration indicated that integration is primarily conducted within the domains of *geometry* and *number operations*, and that it mainly involves CT skills such as problem decomposition, pattern recognition, abstraction, algorithm design, and debugging (Lv et al., 2023). Grounded in both theoretical and empirical considerations, the components included in the current study are *decomposition, algorithm design, abstraction, pattern recognition, debugging, data processing,* and *generalisation* within the epistemological connection between *mathematical thinking* and *computational thinking* (Dinçer Aksoy et al., 2025). Given the role of mathematical modelling problems in fostering mathematical thinking (Kaiser & Sriraman, 2006), the study utilises components included in the related framework.

Moreover, prior syntheses have emphasised that abstraction, in particular, plays a central role in mediating mathematical reasoning and problem solving (Rich et al., 2024; Wing, 2006). Unlike earlier research, the current study conceptualises abstraction in two distinct forms—empirical and reflective—providing a more nuanced lens for the analysis of the participant teachers' cognitive actions (Beth & Piaget, 2013; Cetin & Dubinsky, 2017). While abstraction is discussed in detail, the other components draw upon a synthesised perspective from the literature in order to highlight their function within the modelling process.

Algorithm design refers to the component that involves defining a sequence of steps to solve a problem (Shute et al., 2017) and then analysing the problem-solving procedure itself (Maharani et al., 2019). This action entails breaking the solution process down into various stages so as to create a structured roadmap of the solution. This component therefore includes developing strategies to predict, structure, and implement problem-solving steps using various techniques. In essence, algorithm design represents the ability to create a structured, strategic framework for problem solving (Barr & Stephenson, 2011; Grover & Pea, 2013).

Decomposition is an analytical approach that involves breaking problems down into smaller, more manageable parts or sub-problems (Haşlaman et al., 2024; Kilpeläinen, 2010; Rich et al., 2019; Yadav et al., 2017). This component includes identifying the steps required to solve a particular problem and finding solutions to each step through the application of various methods and strategies (Maharani et al., 2019). Characterised by an analytical perspective, decomposition represents a multidimensional skill that addresses problem solving through systematic analysis, modelling, computation, comparison, and data-driven approaches.

Data processing encompasses the collection, analysis, and visualisation of data (Haşlaman et al., 2024) gathered from various sources (Barr & Stephenson, 2011) such as virtual museums or via the Internet. The collected data or mathematical calculations (e.g., surface area calculations, ratio analyses) are then visualised through graphs and tables (Weintrop et al., 2016), and the data analysed in order to identify any patterns that it may contain Shute et al. (2017). Additionally, the data processing component includes testing and interpreting the accuracy of models created using Internet-sourced data and/or from official statistical sources. In summary, this component represents a skill set that involves collecting, visualising, analysing, and validating data from multiple disciplines in order to construct and refine models.

Pattern recognition is a stage that involves the identification of the structure and patterns of a particular problem and then uncovering any similarities between the existing knowledge and other prior known information (Chen et al., 2023). It also includes serialisation, which represents the ability to logically organise data (e.g., length, weight, age) (Kidd et al., 2012), creating visual and numerical models, categorising data based on similarities, and also construct reasoning based on prior data or knowledge. In essence, pattern recognition represents the ability to observe patterns, trends, and regularities in collected data (Hsu & Hu, 2017).

Generalisation encompasses being able to understand a whole problem based on its inherent patterns, predicting broader contexts, and creating models, rules, principles, or theories within the appropriate context (Shute et al., 2017). This component involves moving from specific observations to more broader structures, examining relationships between distinct and disparate parts, and constructing models based on these relationships. The generalisation stage includes cognitive factors necessary for problem solving (Kallia et al., 2021) and represents a multidimensional skill set that progresses from specific observations to dealing in wider contexts.

Finally, debugging is the process of identifying and correcting errors or deficiencies encountered during the whole problem-solving process. This stage involves predicting potential challenges, identifying and correcting inconsistencies in the solution process, validating data against assumptions and real information, redefining assumptions where appropriate, and making adjustments in order to achieve a near as possible seamless problem-solving experience (Papert, 1980). In essence, debugging represents a skill set that enhances the accuracy and reliability of the solution process by identifying any issues encountered, analysing their respective sources, implementing appropriate corrective actions, reevaluating techniques, validating data, and updating assumptions (Lodi & Martini, 2021).

Building upon the framework outlined in Table 1, the current study conceptualised an integrated perspective that connects computational thinking components with the phases of mathematical modelling. Within this framework, abstraction is able to be examined in two distinct forms: (1) empirical abstraction, which filters out irrelevant details, and (2) reflective abstraction, which enables higher-order generalisation to be realised. The remaining components are positioned as iterative and overlapping processes that help to shape the participant teachers' cognitive actions during the modelling process. This conceptualisation is illustrated in Fig. 2, which visualises how CT components interact dynamically across the modelling cycle.

Fig. 2 presents the proposed conceptual alignment between CT components and the mathematical modelling cycle. The framework conceptualises how CT processes operate across and between phases, sustaining an iterative cognitive system that links real, mental, and mathematical representations. At the outset, Decomposition (2) and Empirical Abstraction (1a) co-function to break down complex, context-rich situations into more manageable subproblems and to isolate essential variables through *subtractive reasoning*. Reflective Abstraction (1b) then reorganises these fragmented elements into structured conceptual models that prepare the ground for mathematical representation. Across the intermediate phases, Algorithm Design (5) governs the sequential organisation of actions, while Pattern Recognition (3), Data Processing (6), and Reflective Abstraction (1b) act as supportive cognitive mechanisms. Pattern recognition enables teachers to detect recurring numerical or spatial relationships, whilst data processing bridges empirical observation and mathematical formulation, and reflective abstraction facilitates continuous restructuring of knowledge as the model evolves.

In the latter phases, Generalisation (4) and Debugging (7) become prominent as learners interpret and validate results. Generalisation allows insights to be extended across analogous contexts, reinforcing the transferability of model structures, while debugging ensures accuracy and coherence through reflective self-regulation. The transition from real results back to the real model symbolises this reflective loop; whereby learners revisit their assumptions, reframe variables, and reconstruct more refined representations of reality. Importantly, these CT components do not unfold in a strictly sequential order; rather, they operate simultaneously and interdependently, forming an overlapping network of cognitive actions. This simultaneity captures the dynamic nature of teachers' computational engagement—where abstraction, decomposition, algorithmic reasoning, and validation continually interact to sustain conceptual coherence throughout the modelling process.

Overall, the framework portrays CT not as a linear set of discrete skills but as a dynamic and recursive cognitive system that underpins the epistemic movement between real, mental, and mathematical worlds within the modelling process. This integrated framework offers a systematic pathway linking the conceptual foundations of CT to its practical implementation within modelling-based teacher education.

3. Method

3.1. Research design and participants

The study employed a design-based research method to design, develop, and implement a TDC for CT-integrated maths teaching through mathematical modelling. Design-based research (DBR) can be utilised to design and develop new learning environments as well as educational practices and theories (Aşık & Yılmaz, 2017; Design-Based Research Collective, 2003; Kuzu et al., 2011).

The study involved four participant secondary school teachers; with three having taught mathematics and one who taught computer science. The selection criteria prioritised those with diverse professional experience and varying levels of expertise in mathematical modelling and CT in order to enrich the study's findings (Creswell & Creswell, 2018; Patton, 2015). The first participant (MT1) was a secondary school mathematics teacher with 2 years of professional experience and was specialised in both mathematical modelling and CT. The second participant (MT2) had 6 years of professional experience, possessed knowledge of mathematical modelling, but had limited familiarity with CT. The third participant (MT3) had 15 years of professional experience but lacked prior

Table 1
Definition of CT components (as applied in the current study).

Component	Definition
Empirical Abstraction	Simplifying problems by focusing on the essential variables and systematically excluding irrelevant aspects.
Reflective Abstraction	Structuring problem situations and developing solution strategies by elevating the level of abstraction.
Decomposition	Breaking complex problems down into smaller, more manageable subproblems for systematic analysis.
Algorithm Design	Defining, managing, or implementing a sequence of steps or procedures in order to solve a problem strategically.
Pattern Recognition	Identifying recurring structures, trends, or similarities across data and contexts.
Generalisation	Extending insights from specific cases to broader contexts, creating rules or models.
Data Processing	Collecting, analysing, and representing data through graphs, tables, or other models.
Debugging	Detecting and correcting errors or inconsistencies within the problem-solving process through self-regulation.

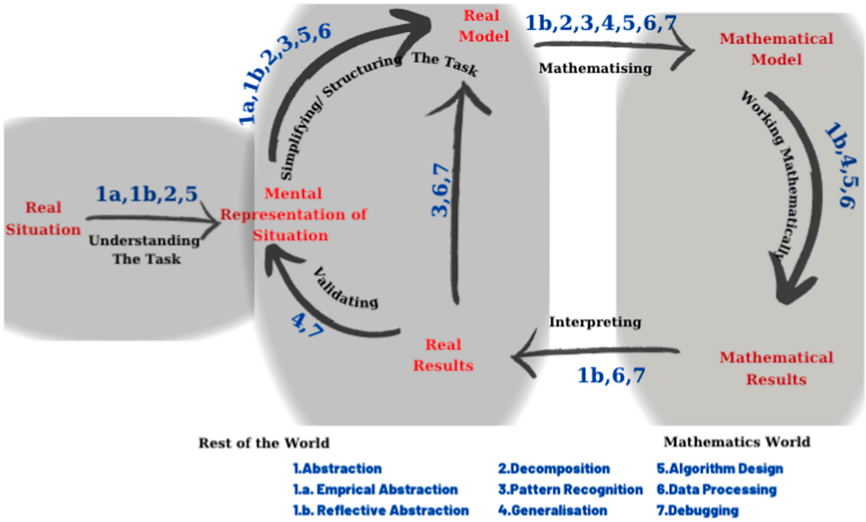

Fig. 2. Conceptual framework illustrating the overlap between CT components and the mathematical modelling cycle.

exposure to both mathematical modelling and CT. The fourth participant (CST) was a computer science teacher with 16 years of professional experience, expertise in CT, and a foundational understanding of mathematical modelling. A coding system (MT1, MT2, MT3, and CST) was used to maintain participant anonymity whilst ensuring clarity in the analysis of the collected data.

3.2. Teacher development course design

The design of the TDC was developed in three phases. First, a comprehensive literature review was conducted in order to examine the relationship between CT and mathematical thinking, followed by a holistic evaluation of the findings. Second, the content was structured into three modules (CT, mathematical modelling, and integration) together with three practice sessions. While the first two modules provided the necessary theoretical foundation, the third focused on integration, with practical sessions that enabled the

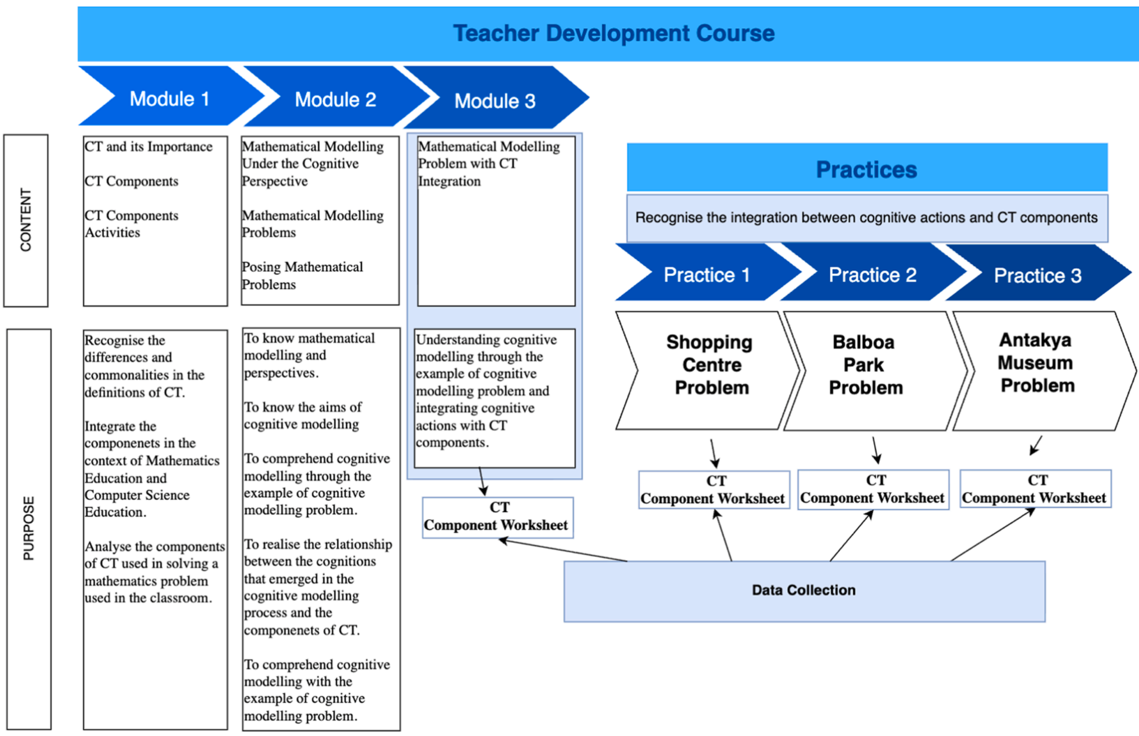

Fig. 3. TDC framework, module contents and purposes, and practices.

participants to apply their newly gained knowledge to different problem situations. The practice sessions were specifically designed to progressively reduce researcher support, allowing the participants to implement the integration more independently at each stage. Finally, expert opinion was collected, and the programme finalised in light of their evaluation. The overall framework of the TDC, including the module contents, purposes, and practices, is illustrated in Fig. 3.

Fig. 3 presents the design of the TDC, illustrating the progression from CT concepts to mathematical modelling and their integration, while also outlining the module contents, purposes, and practices that operationalise this progression within the teacher education context. In addition, the TDC summarises the data collected on the CT-modelling nexus across the modules and practices, highlighting how these components connect and are enacted throughout the programme. The first module, the CT module, was aimed at facilitating teachers to explore CT components and to understand their connection to mathematics education through the use of mathematical problems to guide classroom practice. In the second module, mathematical modelling, warm-up problems based on mathematical modelling aimed to help the participant teachers to solve and comprehend modelling problems more effectively. The third module, integration, was aimed at discussing and examining how the participant teachers approached mathematical modelling problems and to identify and reveal the CT components involved during the problem-solving process. Comprehensive details regarding each module, including their respective objectives, learning outcomes, and pedagogical methodologies are outlined in Table 2. The overarching goal of the developed TDC was to equip the participant teachers with the requisite knowledge, skills, and competencies to integrate CT appropriately into their mathematics education through mathematical modelling.

Table 2
Details about the developed TDC.

Content		Task	Participants Role (Outcome)
Modules (M1-M3)			
M1	Computational Thinking (CT)	Participants solve problems, then group discussion help reveal how CT components emerge during the problem-solving process.	Participants all contributed by working on assigned tasks with opportunities to apply cognitive actions directly within the process.
M2	Mathematical Modelling (MM)	Participants work collaboratively on a modelling problem and examine how their cognitive actions align with the modelling process phases.	Participants all contributed by working on assigned tasks with opportunities to apply cognitive actions directly within the process.
M3	CT Integrated Maths Education Through Mathematical Modelling	Participants address a modelling problem focused on integrating CT components and modelling phases via group discussion.	MT1 and CST's CT modelling experience resulted in their discourse dominance. MT2 and MT3 actively investigated stated hypotheses, executed mathematical operations, derived results and verified them to online resources. With researcher guidance, the participants worked collaboratively to identify overlaps between cognitive actions emerging during the modelling process and CT components.
Practices (P1-P3)			
P1	Shopping Centre	Participants work collaboratively to solve a mathematical modelling problem (see Appendix 2), then reflect and discuss how CT components and modelling phases are integrated.	MT1, MT2, and MT3 used average adult male height in Türkiye, engaged in data collection, analysis, and verification against real-world observations during data processing. Following the CST's suggestion, the pattern unit was revised (different reference individual used) to account for photographic perspective. Under the researcher's guidance, the participants collaboratively identified overlaps between cognitive actions that emerged during the modelling process and CT components.
P2	Balboa Park	Participants work collaboratively to solve a mathematical modelling problem (see Appendix 2) while identifying, articulating, and integrating CT components based on the phased modelling approach.	MT1's prior modelling experience, combined with CST's familiarity with daily-life contexts, helped enrich the range of CT components identified. Building on this expertise, the participants collaboratively constructed new problem scenarios inspired by the first two tasks and developed related mathematical models, further demonstrating their capacity to apply CT components in varied contexts.
P3	Antakya Museum	Participants work collaboratively to solve a mathematical modelling problem (see Appendix 2) while identifying, articulating, and integrating CT components based on the phased modelling approach.	All participants actively contributed to simplifying the task by selecting a section of the image as a unit, then applying pattern recognition to identify repeating structures (see Fig. 11a-c). Together, they attempted to extend the unit to the whole mosaic, an abstraction process that they did not always recognise as an intertwined cognitive action. During model construction, they employed generalisation by choosing the trapezium as a geometrically hierarchical and broadly applicable structure. Through these collective actions, the participants not only applied CT components more effectively but also demonstrated progress in articulating the cognitive processes guiding their modelling decisions.

3.3. Teacher development course implementation

Implementation of the developed TDC was organised through structured modules and practices, with each designed to link CT components with the specific phases of mathematical modelling. Table 2 summarises the implementation of the TDC which followed a structured progression through three modules and three practices, as well as detailing each module and the individual practices, tasks, and participant roles involved.

In Module 1 (CT), the participants solved problems and examined how CT components emerged during the problem-solving process through group discussion, thereby making their cognitive actions more explicit. Fig. 4 summarises the instructional tools employed in Module 1. The researchers employed various instructional tools in Module 1: a concept map to elucidate Piaget's theory of abstraction (see Fig. 4a), a PowerPoint presentation delineating components and examples (see Fig. 4b), and a question to gauge cognitive processes elicited during problem solving (see Fig. 4c).

The researchers prepared an infographic (see Fig. 4a) to illustrate Piaget's theory of abstraction and shared it with the participants. The participants then engaged with examples of each CT component in everyday, mathematical, and interdisciplinary contexts in order to construct their understanding (see Fig. 4b). In the final step, the participant teachers worked with a classroom problem based on their own practice of teaching maths, demonstrating how CT components emerged during problem solving and applying the principle of 'From the known to the unknown principle' (see Fig. 4c).

Module 2 (Mathematical Modelling) centred on solving modelling problems and reflecting on how cognitive actions aligned with modelling phases; again, with active contributions from all of the study's participant teachers. Borromeo Ferri's (2006) mathematical modelling cycle served as the foundational framework for Module 2, which was introduced and explained in detail to the participant teachers. Both the researcher and the participants then solved mathematical modelling warm-up problems, Borromeo Ferri's (2014) bed problem, and Fermi's problems to apply the acquired concepts.

Module 3 (CT Integrated Mathematics Education Through Modelling) emphasised the integration of CT components and modelling phases, where the participant teachers, guided by the researcher, collaboratively identified overlaps and complementarities across

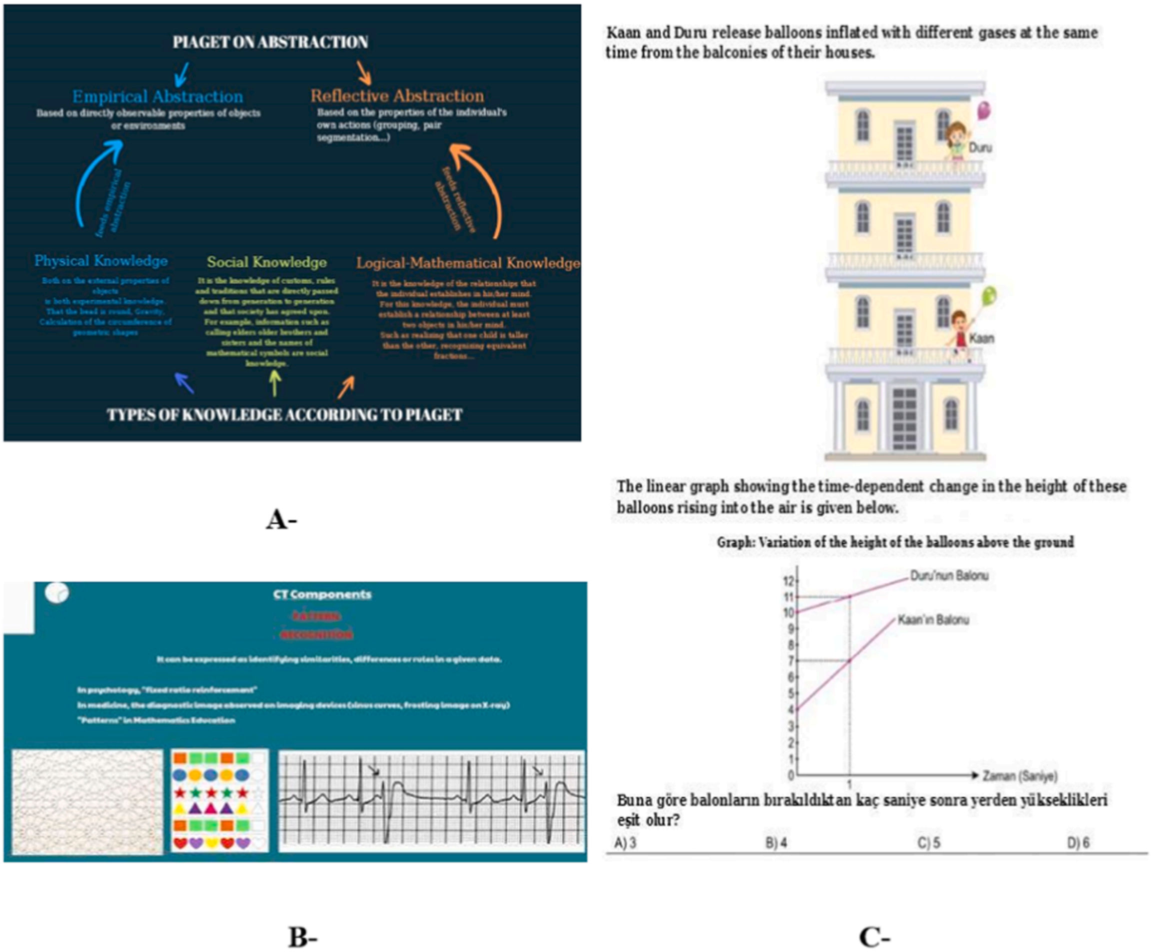

Fig. 4. Illustrative materials from Module 1 of the TDC.

these domains. The teachers were presented with the Straw Bale problem, and then they progressed through the following steps: (1) solving the modelling problem; (2) drawing connections between the cognitive processes observed and Borromeo Ferri's (2006) modelling process; and (3) integrating these cognitive processes into CT components, which were documented through the completion of a CT component worksheet.

Across all three modules, the researcher consistently offered guidance and support to the participant teachers, providing more intervention during the early stages and gradually reducing the input needed as the participants gained confidence with each task.

The practices were designed to extend and consolidate this integration in real-world contexts. In the Shopping Centre task, the participants were engaged in both the data collection and analysis, activating the data processing component while revising their approach through pattern recognition. In the Balboa Park practice, the interplay of the participants' prior experience enabled them to construct new scenarios and models that demonstrated the flexible application of CT components. Finally, in the Antakya Museum practice, the participants employed abstraction and generalisation by identifying repeating mosaic structures and extending them to the whole design, showcasing progress in their being able to articulate the cognitive processes behind each modelling decision.

In total, the three modules lasted for 6 h and 23 min, and together with the three practices, the entire process was completed in 11 h and 32 min.

3.4. Data collection tools

Data collection was conducted utilising three main instruments: (1) the CT Components Worksheet, where participants documented CT components observed during Module 3 and the three practice sessions; (2) the CT Component Worksheet, where participants solved the mathematical modelling problems during lesson designs; and (3) video recordings of the TDC sessions designed to capture real-time implementation and reflective practices. These instruments were systematically applied across Module 3, focusing on three iterative practice sessions (Practices 1–3) for the integration of CT. With the application of these tools, the study's data were systematically collected from Module 3 and from Practices 1–3 to monitor the participant teachers' progress in their integration of CT and mathematical modelling.

Table 3 was developed in order to evaluate how the participant teachers integrated cognitive modelling stages and CT components during Module 3 and the three subsequent practice sessions. Cognitive processes were identified by the participant teachers as they deliberated on the integration of CT components, and the appropriate details. An example CT Component Worksheet is presented in

Table 3
CT components worksheet for the Straw Bale problem.

Modelling Cycle – Cognitive Perspective	Example	CT Component
Understanding Task	D: 'Positioning of straw bales decreases from bottom to top (five at the bottom and one at the top), and solving the problem gradually step by step'. EA: 'Importance of human height to the solution'. RA: 'Relating straw bale radius to human leg length'.	D: Decomposition EA: Empirical Abstraction RA: Reflective Abstraction
Simplifying / Structuring Task	EA: 'Ignoring pressure of straw bales against each other'. D: 'Analysing average human leg length according to straw bales'. DP: 'Investigation of human height and leg length'. AD: 'Building a solution plan as an algorithm'.	EA: Empirical Abstraction D: Decomposition DP: Data Processing AD: Algorithm Design
Mathematising	RA: 'Deciding what information is needed to solve the problem (e.g., knee angle, ratio between leg length and height, equality of lower and upper leg length, $30^\circ-30^\circ-120^\circ$ triangle, Pythagorean theorem, tangent circles, etc.)'. D: 'Calculating human leg length, isosceles triangle side, radius of a straw bale according to isosceles triangle, and bale cluster height according to bale radius'. DP: 'Investigation of a young woman's height and leg length'. G: 'Use of Pythagorean theorem, the $30^\circ-30^\circ-120^\circ$ triangle [referring to angles] and tangent circle properties to draw conclusions'. PR: 'Continuing bale positioning in the mind to complete the visual'. AD: 'Building the mathematical model first, then designing which mathematical operations to perform'.	RA: Reflective Abstraction D: Decomposition DP: Data Processing G: Generalisation PR: Pattern Recognition AD: Algorithm Design
Working Mathematically	G: 'Use of Pythagorean theorem to construct a model for different human heights'. RA: 'Operations performed based on mathematical knowledge'. D: 'Calculating human leg length, isosceles triangle side, straw bale radius according to isosceles triangle, and height of straw bale cluster according to bale radius'. AD: 'Step-by-step implementation of agreed actions'.	G: Generalisation RA: Reflective Abstraction D: Decomposition AD: Algorithm Design
Interpreting	DP: 'Noted that the collected data overlapped with daily life'. Deb: 'Checked for any bugs'.	DP: Data Processing Deb: Debugging
Validating	G: 'Accepting solution accuracy as overlaps with real-life data and the possibility of converting to different heights'. Deb: 'Checking solution accuracy to knee position, leg length, and knee angle'.	G: Generalisation Deb: Debugging

Table 3.

The participants each provided their consent for both audio and video to be recorded throughout the various sessions of the study, which were utilised in order to prevent data loss in real-time settings and to facilitate thorough analysis as the researchers were able to revisit the sessions as needed.

3.5. Data analysis

Analysis was conducted according to a qualitative framework that combined Borromeo Ferri's (2006) modelling cycle, viewed from a cognitive perspective, with the CT components defined in the current study. This dual lens allowed the researchers to trace how the participants' cognitive actions unfolded across the modelling phases while simultaneously capturing the integration of CT.

To ensure inter-coder reliability, Cohen's Kappa coefficient (Cohen, 1960) was employed; a statistical measure used to evaluate the degree of agreement between two independent raters who classify data into categorical levels. The Kappa statistic is restricted to comparisons between two coders and assumes that both the coded items and the raters' judgments are applied independently (Brennan & Prediger, 1981). In the current study, one researcher's coding was independently re-coded by a second researcher, and the level of their agreement calculated using the Kappa coefficient. The resulting Kappa value was 0.83, indicating a very high level of consistency between the coders.

Multiple data sources were utilised (worksheets, audio recordings, and video recordings) in order to strengthen the validity of the interpretations made, and the participants anonymity and confidentiality of their actions and responses were protected through the assignment and use of pseudonyms throughout the study and in its reporting. The study participants were also invited to reflect on both the practices they undertook and the subsequent analysis, thereby reinforcing confirmability and ensuring that their discourses remained visible in the data.

The first author led the development of the study's educational content, as well as being responsible for participant engagement and the facilitation of each session, initially offering a greater level of guidance and then gradually withdrawing their support as the participants became more able to take on a greater level of responsibility themselves. The study's other authors enriched the process by contributing to the course design and by evaluating it independently, having not been directly involved in its implementation.

Finally, so as to ensure transparency and reproducibility, a coding table was created based on the CT Component Worksheet (see Appendix 1).

4. Findings

4.1. Emerging CT components during mathematical modelling in Module 3

In Practice 1, the participants completed the modelling cycle through group discussion. Table 3 provides a summary of the overlaps between the participants' identified cognitive actions during the modelling phases and the corresponding CT components. The modelling phase unfolded progressively, with the participant teachers moving from initial problem comprehension toward validation of their solutions. At the outset, decomposition was evident as the participants positioned the straw bales on a step-by-step basis, reflecting a breaking down of the complex problem into more manageable subproblems. This was supported by empirical abstraction, where they simplified the problem by focusing on essential variables such as human height while disregarding bale pressure within the stacked formation. Reflective abstraction emerged when the participants related bale radius to human leg length, illustrating how higher-order conceptualisation helped them structure the problem through real-world features. Fig. 5 provides examples of the participants' sketches and solution strategies that illustrate these steps.

Fig. 5. Participants' on-screen representations and solutions for the Straw Bale Problem.

During the simplifying and structuring phase, the participants employed empirical abstraction by eliminating irrelevant forces, while decomposition again supported the analysis of human leg length relative to straw bale dimensions, which they illustrated in red in Fig. 5 based on the human figure's leg. Data processing facilitated evidence-based reasoning with regards to human body proportions, and algorithm design was evident as the participants began to define a sequence of steps in their preliminary solution plan.

In terms of mathematisation, reflective abstraction became central as the participant teachers selected the necessary mathematical properties, including triangle relations, circle tangency, and Pythagorean theorem. Decomposition and data processing guided the systematic analysis of the straw bale dimensions, while pattern recognition enabled the participants to identify recurring structures in the bale positioning. Generalisation was evident when the participants applied mathematical theorems to broader configurations, and algorithm design consolidated these actions into a coherent model.

Working mathematically deepened these processes, with calculations implemented on a step-by-step basis through algorithm design, supported by decomposition, while reflective abstraction integrated mathematical knowledge into the solution. Generalisation extended the model to different heights, showing transfer from particular to general structures.

Finally, in interpreting and validating the results, data processing was used to compare calculations with real-world data, and debugging enabled the participants to detect and correct any inconsistencies. Validation was strengthened through generalisation, confirming the adaptability of the model, and through debugging, which refined assumptions about human knee angle and leg-length ratios.

Across the modelling cycle, the participants demonstrated a progressive orchestration of CT components. Each phase reflected distinct yet interconnected practices that supported the transition from problem understanding to solution validation. As revealed in Fig. 6, this progression can be visualised through the unfolding of cognitive actions across the modelling process, highlighting overlaps between the participants' identified cognitive actions within the modelling phases and the corresponding CT components. The cycle began with decomposition and empirical abstraction, advanced toward reflective abstraction and generalisation in mathematisation, and extended into validation through data processing and debugging. Each stage revealed how CT practices intertwined to structure systematic problem solving within a real-world context.

4.2. Emerging CT components during mathematical modelling in Practice 1

During Practice 1, the participants completed the modelling cycle through group discussion. Table 4 summarises the overlaps between the participants' identified cognitive actions in the modelling phases and the corresponding CT components. The modelling phase progressed as the participants attempted to estimate the height of the shopping centre from a photograph taken inside the building. During the initial phase of understanding the task, reflective abstraction was central. The participant teachers assumed a potential relationship between people visible in the photograph and the building's height, deciding to use the average male adult human height in Türkiye as a strategic reference point. This early choice shows how abstraction provided a cognitive anchor, allowing the group to structure the problem in terms of familiar and measurable features.

In simplifying and structuring the task, participants began to formalise their approach. Empirical abstraction appeared as they took multiple measurements of the same human individual in the image (see Fig. 7a), averaging these to establish a unit of measure. Decomposition was evident in the planning of separate steps for comparing average human male height in Türkiye and building heights, while algorithm design guided the creation of a clear roadmap for measurement, calculation, and interpretation. Data processing supported this phase by incorporating average height statistics for Turkish adult males retrieved from online sources. Together, these practices illustrate the transition from intuitive observations to a structured problem-solving plan.

The mathematising phase introduced more formal reasoning, with the participants' investigation of 'average adult male height data

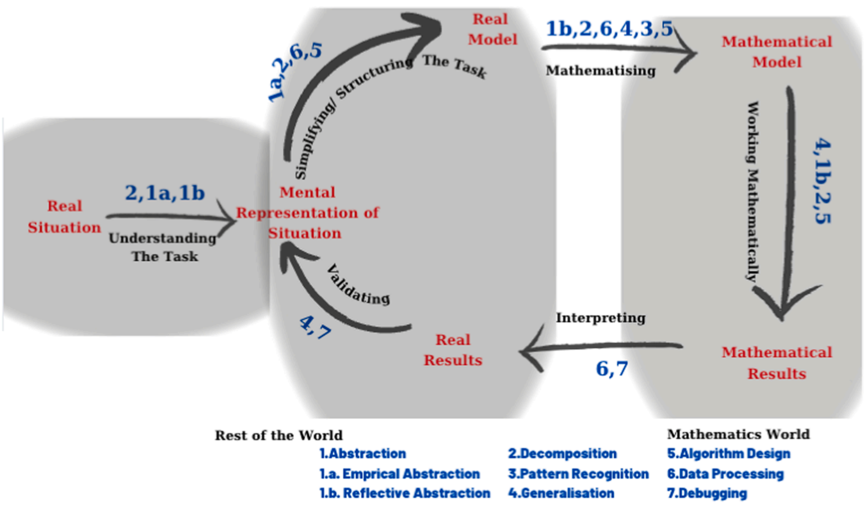

Fig. 6. CT components emerged from the Straw Bale Problem.

Table 4
CT components worksheet for the Shopping Centre Problem.

Modelling Cycle – Cognitive Perspective	Example	CT Component
Understanding the Task	RA: 'Potential relationship assumed between people in the photograph and the building height'. RA: 'Using the height of a person as an important strategic factor instead of tree height is an abstraction'.	RA: Reflective Abstraction
Simplifying/ Structuring the Task	EA: 'Taking different measurements of the same height [person's height taken as unit] and averaging' and 'Deciding to take the selected person as the average adult male height in Türkiye'. D: 'Planning separately to measure different heights, determine the ratio between them, and to create a formula for the actual ceiling height'. AD: 'Creating a roadmap in the form of planning, measuring, calculating, and interpreting the solution process'. DP: 'Finding the average adult male human height on the Internet'.	EA: Empirical Abstraction D: Decomposition AD: Algorithm Design DP: Data Processing
Mathematising	DP: 'Finding the average adult male human height online'. AD: 'Implementing the first part of the determined algorithm'. Deb: 'Deciding to recalculate according to the perspective angle'. PR: 'Height of one storey before calculating the height of numerous storeys to determine the ratio between building height and human height'. D: 'First, find the ratio between the height of the building and the human height, then reach the result from the actual human height'. G: 'Finding that there is a 20-fold ratio between human height and building height'. RA: 'Recognising the relationship between the selected human height of 0.7 cm [human height length in the yellow box in Fig. 6b and the determined ceiling height of 14 cm'.	DP: Data Processing AD: Algorithm Design Deb: Debugging PR: Pattern recognition D: Decomposition G: Generalisation RA: Reflective Abstraction
Working Mathematically	G: 'A: Shopping centre height measured from image, B: Human height measured from image, C: Creating $A \div B \times C$ model for the average height of an adult male human in Türkiye', and 'applying 20-fold ratio for average adult male height in Türkiye'. RA: 'Ceiling height 20 times the height of an average human'. D: 'Stages of dividing ceiling height by human height and then multiplying result by average human height'. AD: 'Sequential execution of these operations'.	G: Generalisation RA: Reflective Abstraction D: Decomposition AD: Algorithm Design
Interpreting	DP: 'Checking whether the mathematical result was consistent with the actual result'. Deb: 'Obtaining data through research to determine how much the findings corresponded to real-life information'. G: 'Teacher 1: Shopping centre height measured in the image, Teacher 2: Human height measured in the image, Teacher 3: Average height of an adult human male in Türkiye. $A \div B \times C$ using the model to describe different heights'.	DP: Data Processing Deb: Debugging G: Generalisation
Validating	G: 'Checks were made at each step of the solution, and it was decided that the real-life data and mathematical results were consistent'. Deb: 'The height of another shopping centre can be predicted using this result'.	G: Generalisation Deb: Debugging

a b

Fig. 7. Participants' on-screen representations and solutions for the Shopping Centre Problem.

in Türkiye' entailing data collection, analysis, and verification against real-world observations having exemplified the data processing component. At the CST's suggestion, the participants modified the pattern unit during the mathematisation phase, shifting from the individual in Fig. 7a to that shown in Fig. 7b due to photographic perspective. Debugging was applied when they adjusted calculations to account for perspective distortion in the photograph. Pattern recognition surfaced as they estimated the height of a single storey before extrapolating to the full height of the building, while decomposition helped in sequencing the ratio-based comparisons. Generalisation became visible when the participants concluded that the building was approximately 20 times the height of one adult human male, and reflective abstraction supported recognition of proportional relationships between measured and real-life values. The participants also clarified that Internet-based research was treated as debugging since it served primarily to identify potential errors rather than process data. These interwoven CT practices enabled the development of a coherent mathematical model grounded in both visual data and external references.

In working mathematically, the participants formalised their model ($h = A \div B \times C$): the measured human height (B) was related to the measured building height (A), and scaled using the average height of an adult male in Türkiye (C) to estimate the total building height. Generalisation was evident in applying the 20-fold ratio, reflective abstraction in articulating the proportional link between average human height and ceiling height, decomposition in dividing and scaling operations, and algorithm design in sequentially executing the calculations. This stage highlights how multiple CT components converged to operationalise the model.

During interpreting, the participants drew upon data processing to check the consistency of their mathematical outcomes with real-life expectations. Debugging was present as they sought additional sources to validate the plausibility of their findings. Generalisation appeared in the way that the participant teachers expressed the model's formula as a transferable tool for estimating different heights, demonstrating their ability to abstract beyond the immediate task.

Finally, in validating, the participants compared their results with external data and confirmed consistency between their mathematical model and actual building dimensions. Generalisation allowed them to extend their method to predict the heights of other shopping centres, while debugging ensured the robustness of their assumptions and calculations. This final stage helped to illustrate the iterative interplay of CT practices, where accuracy was continuously checked against empirical reality.

This session demonstrated that interdisciplinary collaboration significantly enhanced the participants' ability to identify overlaps between CT components (e.g., abstraction, algorithmic design) and mathematical modelling processes. Analysis of group discourse revealed improved intra-group communication, accelerating the integration of CT concepts such as data processing and debugging into problem-solving workflows. The participants exhibited increased proficiency in solving modelling problems, with cognitive processes and CT component application growing in tandem. However, articulating internal cognitive mechanisms during modelling remained challenging, necessitating the researcher to provide guidance to externalise implicit reasoning. As MT2 noted, *'Realising what you do while solving the problem can be more difficult than solving it',* underscoring the complexity of metacognitive awareness in CT-integrated modelling. Fig. 8 visually synthesises these dynamics, illustrating the interplay between emergent cognitive actions and CT components during the 'Shopping Centre Problem' solution process. These findings highlight the dual role of structured collaboration and visual scaffolding in fostering CT-mathematics integration, while emphasising the need for targeted support to bridge gaps in metacognitive articulation.

Fig. 8 visualises the overlaps between the participants' identified cognitive actions in the modelling phases and the corresponding CT components. Across the modelling phases, the participants consistently activated CT components in an iterative manner. Reflective and empirical abstraction provided the foundation for structuring the problem, decomposition and algorithm design ensured systematic progression, while data processing and pattern recognition supported mathematising. Generalisation and debugging played crucial roles in interpreting and validating results, illustrating how CT practices intertwined to transform a real-world visual estimation task into a mathematically reasoned solution.

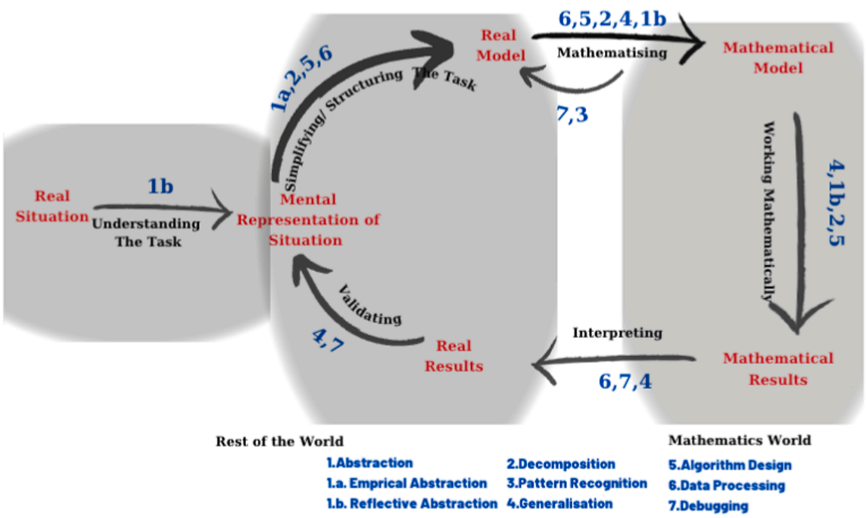

Fig. 8. CT components that emerged from the Shopping Centre Problem.

4.3. Emerging CT components during mathematical modelling in Practice 2

In Practice 2, the participants completed the modelling cycle by way of group discussion. Table 5 summarises the overlaps between the participants' identified cognitive actions in the modelling phases and the corresponding CT components. In addressing the challenge of 'capturing a correct perspective because a difference in perspective would give the wrong result', the participants avoided further errors by prioritising accurate perspective selection, which they explicitly categorised as debugging. During Practice 1, their initial neglect of perspective necessitated redefining the pattern unit, an adjustment which they later applied in Practice 2. As the participants gained experience in CT-integrated modelling processes, they became more proactive with practices aimed at error anticipation and prevention.

The modelling phase began with understanding the task**,** where reflective abstraction guided the participants' decision to use railing length rather than a palm tree as the reference point. Having learnt from the Straw Bale problem, the participants employed a process of writing and modelling problem assessment in order to calculate the height of the shopping centre depicted in the image. In the visual given in Practice 2, MT1 said, *'Let's not calculate length again this time, but let's differentiate the problem instead'*. This led to the writing and solving of a modelling problem regarding how much paint would be needed to paint the wall seen in the visual. Algorithm design was evident as they mentally structured the stages of dividing the surface into smaller units to calculate the total area, while decomposition supported their plan to compute the surface area and paint volume needed in separate steps. These early actions illustrate how the participants quickly transformed an ill-structured task into a sequence of solvable subtasks.

During simplifying and structuring the task, MT1 suggested they perform a Google image search. From their search they decided that the structure was Balboa Park, since MT1 found views of Balboa Park from different angles. The participants preferred the image shown in Fig. 9 as the most appropriate visual representation of the data they needed as additional mathematical information. The participants applied empirical abstraction by ignoring ornamental details such as the marble capitals and treating complex shapes (e.g., the dome) as simplified geometric forms. Reflective abstraction enabled them to adopt railing length as a unit of measurement. The CST said*, 'I am looking into whether there is a standard for guardrail spacing'*, with data processing thereby providing real-world standards for railing dimensions. Decomposition was visible as the building was divided into squares, semicircles, and rectangular prisms, and

Table 5
CT components worksheet for the Balboa Park problem.

Modelling Cycle – Cognitive Perspective	Example	CT Component
Understanding Task	RA: 'Deciding to use railing lengths instead of palm trees'. AD: 'The formation of the stages in the mind helped visualise that the total area could be found by dividing the surface'. D: 'Forming the idea that the area calculation could be calculated from separately calculating the area and the litre amount'.	RA: Reflective Abstraction AD: Algorithm Design D: Decomposition
Simplifying/ Structuring Task	EA: 'Ignoring the building's marble capitals and thinking of them as flat, considering the dome as a half circle, ignoring the ceiling, examining the image of the building from a different angle'. RA: 'The railing was taken as a reference [unit]'. DP: 'Standards of railing lengths found on the Internet; determination of critical lengths'. AD. 'Determining which lengths could be calculated by referring to each other; determining the calculation order of the fields'. D: 'Dividing the whole space into small squares, semicircles, and rectangular prisms'. PR: Participant comment, 'The regions divided into these small squares and semicircles are pattern recognition'.	EA: Empirical Abstraction RA: Reflective Abstraction DP: Data Processing AD: Algorithm Design D: Decomposition PR: Pattern Recognition
Mathematising	RA: 'Taking into account the difference in perspective in calculating the front and rear heights [referring to the measured height] and rounding the values to a close value'. D: 'Finding the heights separately'. DP: 'Distance between guardrails obtained from the Internet'. Deb: 'Capturing a correct perspective because a difference in perspective would give the wrong result'. PR: 'Recognising repeating parts of the surface area' [pattern formed by the distance between two columns]. AD: 'Determining the order of finding the heights'.	RA: Reflective Abstraction D: Decomposition DP: Data Processing Deb: Debugging PR: Pattern recognition AD: Algorithm Design
Working Mathematically	G: 'Producing formulae, using ratio and proportion, finding areas of geometric shapes'. RA: 'Deciding which variables to use in which formulae'. D: 'Separately calculating the values of the equation variables'. AD: 'Implementation of the solution plan'. Deb: 'Detecting and correcting errors when creating formulae'. DP: 'Obtaining information from the Internet on how to create formulae' [e.g., how many m^2 can be painted with 1 L of paint].	G: Generalisation RA: Reflective Abstraction D: Decomposition AD: Algorithm Design Deb: Debugging DP: Data Processing
Interpreting	Deb: 'Examining consistencies between mathematical results and real-life values is debugging'. DP: 'Obtaining real-life values from the Internet is data'.	Deb: Debugging DP: Data Processing
Validating	G: 'Noting consistencies between mathematical results and real-life values'. Deb: 'Concluding that the obtained result [mathematical model] could paint the structure shown in the figure'.	G: Generalisation Deb: Debugging

a	b

Fig. 9. Participants' on-screen solutions for the Balboa Park Problem.

pattern recognition surfaced when the participants identified repeating structures within these divisions. Collectively, these actions highlight how simplification and patterning enabled the participants to create a workable mathematical representation of the building. At this stage, components rich in types of abstraction emerged, with both the CST and MT1 having contributed to this emergence.

In the mathematising phase, MT3 said, *'Length 2 and length 3* [as shown in Fig. 9a-b] *are the same. It looks because different of the visuals, so we need to measure these lengths and convert them to each other'*. Reflective abstraction supported adjustments for perspective distortions, while decomposition guided the separate calculation of multiple heights. The lengths were measured by task sharing within the group. Data processing drew upon online resources to refine distance values, and debugging was seen as crucial in recognising that inaccurate perspective would in fact produce erroneous results. MT2 said, *'We know the length of the guardrail, so we can*

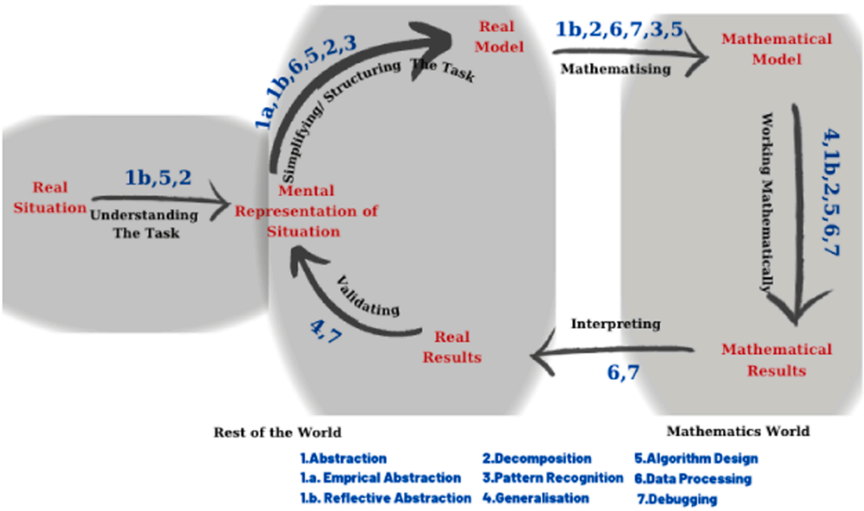

Fig. 10. CT components that emerged from the Balboa Park Problem.

use that as the unit', and the group accepted this suggestion. Pattern recognition reappeared as the participants noted repeating architectural units, and algorithm design helped sequence the height calculations. This stage illustrated how CT practices provided both corrective and constructive mechanisms for building a viable mathematical model.

Length 2 and length 3 in Fig. 9a appear to be the same in everyday life, but have visually different lengths due to perspective. As such, they were measured separately. The length of the railing in yellow box number 1 (see Fig. 9a) was found on the Internet by the CST and the group accepted it as a unit. The proportional relationship between the lengths inside boxes 1, 2, and 3 were then calculated and the participants wrote the calculated values on the visual (see Fig. 9b).

Working mathematically involved formal implementation. The participants created three different mathematical models for the amount of paint needed (in litres) to paint a known area. Generalisation was demonstrated in producing formulae based on ratio and proportion, while reflective abstraction guided the selection of appropriate variables. Decomposition supported the calculation of individual components, and algorithm design structured the execution of operations. Debugging was again visible as the participants detected and corrected errors in their formulae, while data processing extended the model by incorporating real-world information (e.g., data on paint coverage per litre). These practices together show how systematic CT engagement enabled the participants to operationalise their model with some degree of accuracy.

With regards to interpreting, the participants relied upon data processing to retrieve real-life values and on debugging to compare

Table 6
CT components worksheet for the Hatay Archaeology Museum problem.

Modelling Cycle – Cognitive Perspective	Example	CT Component
Understanding Task	EA: 'Completing the original part of the mosaic in the image'. DP: 'Examining the image, identifying the original and non-original parts'. D: 'Each of these stages'.	EA: Empirical Abstraction DP: Data Processing D: Decomposition
Simplifying/ Structuring Task	DP: 'Searching the Internet for better angle photos of the same structure'. EA: 'As a result of research, it was decided that the photograph in the problem situation was better'. RA: 'Deciding to reach the total number of mosaics from the number of mosaics in a specific cross-sectional area by making a proportion'. PR: 'Deciding whether to fill the mosaic floor as pictured or the entire mosaic floor'. [Participants decided between calculating the number of mosaics in the frame of the mosaic work and calculating the number of mosaics in the entire work]. RA: 'It was decided to use the area of the rectangle'. D: 'Planned process steps'. AD: 'It is the solution in line with the planned process steps'.	DP: Data Processing EA: Empirical Abstraction RA: Reflective Abstraction PR: Pattern Recognition D: Decomposition AD: Algorithm Design
Mathematising	EA: 'Fragmentation of the photograph using MS Word gridlines to form ground squares by making the photograph transparent'. DP: 'One-unit square was reached' (see Fig. 11a). RA: 'Discussing that the problem can be solved with length, area, ratio, and proportion calculations'. PR: 'Identifying the repeating trapezium part while segmenting the area and deciding that it is easier to reach the whole by finding the area of that part' (see Fig. 11b). Deb: 'In the process of calculating the area, thinking that a trapezium would be more appropriate (see Fig. 11c) and giving up using the area of a rectangle' (hierarchy of quadrilaterals). G: 'Discussing and deciding that using the area of a trapezium is more generalisable for mathematical model building' (hierarchy of quadrilaterals). D: 'Each step'. AD: 'The entire solution process'.	EA: Empirical Abstraction DP: Data Processing RA: Reflective Abstraction PR: Pattern Recognition Deb: Debugging G: Generalisation D: Decomposition AD: Algorithm Design
Working Mathematically	DP: 'It was determined that there were 32 stones in a unit square. To find the entire area, we counted the length of a side to find how many square units of the area it has'. PR: 'Identifying the repeating rectangular area while dividing the area'. RA: 'We discovered that it would be easier to reach the whole part by finding the area of that part'. G: 'We used the area of the trapezium, not the rectangle, to eliminate the error due to perspective'. AD: 'Applied sequence of operations'. D: 'Sub-operations in algorithm steps'. G: 'Creating a mathematical model, that is, using the variables'. [a, b, and h are shown in Fig. 11.] $\left[(a+b)\cdot\frac{h}{2}\cdot c\right]$	DP: Data Processing PR: Pattern Recognition RA: Reflective Abstraction AD: Algorithm Design D: Decomposition G: Generalisation
Interpreting	EA: We discussed consulting the architect's friend for verification'. DP: 'Research was conducted on mosaics and mosaic museums on the Internet'. D: 'When calculating the area of a parallelogram as if it were a rectangle, it was interpreted that the areas we included and excluded would balance each other out' [The part that the participants refer to as the taken and not taken area while calculating the area of the parallelogram is the case used in the proof of the parallelogram area]. MT1: 'There is also abstraction here since we assume it will balance out'. Deb: 'The trapezium was decided for the most valid area account'.	EA: Empirical Abstraction DP: Data Processing D: Decomposition Deb: Debugging
Validating	G: 'Examining the images in the Antakya mosaic museum, it was discussed whether or not the number of mosaics in rectangular photographs could be reached using the same mathematical model'. Deb: 'As a result, it was decided that this would be more inclusive as the area of the trapezium could also be calculated from the area of a rectangular region'.	G: Generalisation Deb: Debugging

mathematical outcomes with practical feasibility. Finally, in validating, generalisation was used to establish consistency between mathematical and real-world results, while debugging ensured that the derived model was able to reliably predict whether or not the paint quantity was sufficient to cover the surface area. This phase highlighted the iterative interplay of CT practices, where validation was anchored in both abstraction and empirical reference.

Fig. 10 visualises the overlaps between the participants' identified cognitive actions in the modelling phases and the corresponding CT components. Throughout the modelling phases, the participants engaged in a layered use of CT practices: reflective and empirical abstraction anchored problem structuring, decomposition and algorithm design provided systematic scaffolding, while data processing, pattern recognition, and debugging ensured accuracy and adaptability. Generalisation tied these practices together by enabling the model to be transferred to real-world applications, demonstrating the integrative function of CT within complex modelling tasks.

4.4. Emerging CT components during mathematical modelling in Practice 3

In Practice 3, one of the participants (MT3) suggested that the group used a photo he had taken while visiting the Hatay Archaeology Museum. The photograph showed a mosaic work of which the missing parts had been completed with a drawing. The participants agreed to use this artefact to write and solve modelling problems and to build the integration between cognitive processes and CT components. Table 6 summarises the overlaps between the participants' identified cognitive actions in the modelling phases and the corresponding CT components. The modelling phase began with understanding the task. The participants initially applied empirical abstraction to distinguish between the original and non-original parts of the mosaic in the photograph. They engaged in data processing by closely examining the image and searching online for alternative perspectives, while reflective abstraction guided their reasoning as they framed the idea of estimating the total number of tiles from a cross-sectional sample through proportional reasoning. At this stage, pattern recognition surfaced as they discussed whether or not to calculate only the framed mosaic portion or extend their calculations to the entire floor. Supported by decomposition and algorithm design, the participants planned the solution as a sequence of manageable steps, demonstrating how CT practices offered an initial structure for addressing the ill-defined problem.

During simplifying and structuring the task, the participants further enacted empirical abstraction by disregarding unnecessary architectural details such as capitals and domes and reducing them to simplified geometric forms. Reflective abstraction shaped their decision to use railing length as a unit of measure, while data processing was evident in their effort to identify critical measurements from external resources. Through decomposition, they divided the surface into geometric elements such as squares (see Fig. 11a) and rectangles (see Fig. 11b), and by means of pattern recognition they identified repeating motifs across the mosaic. Algorithm design consolidated these steps into a coherent sequence that structured subsequent calculations.

The mathematising phase introduced more formal reasoning. The participants used MS Word gridlines to fragment the mosaic photograph into transparent unit squares, enacting empirical abstraction (see Fig. 11c), while simultaneously retrieving real-world measures of tile units through data processing. Reflective abstraction was visible as they debated whether or not length, area, or ratio-based calculations were the most appropriate, and pattern recognition supported the identification of trapezium repetitions within the design (see Fig. 11c). When the participants recognised that rectangle-based area calculations led to perspective errors, they turned to debugging, abandoning their earlier approach. At this juncture, they employed generalisation by selecting the trapezium as a

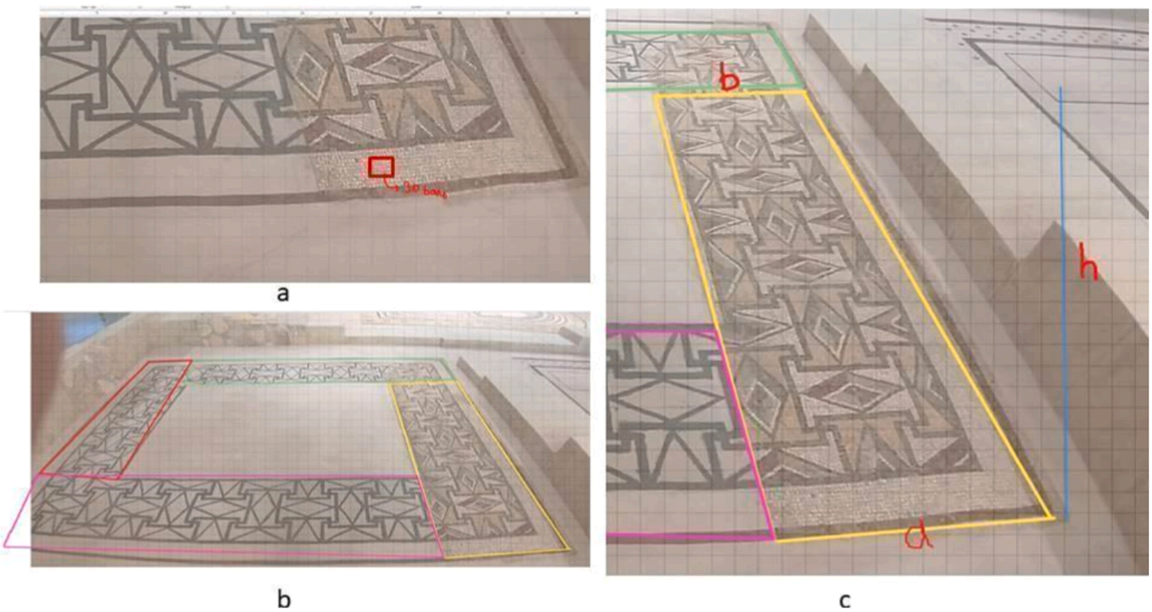

Fig. 11. Participants' work on the mosaic image during the Mosaic Problem.

geometrically, hierarchical, and broadly applicable structure for model construction. This decision not only enhanced accuracy but also provided a transferable mathematical basis for further reasoning. Decomposition and algorithm design ensured that the sequence of calculations remained both systematic and iterative.

In working mathematically, the participants continued this structured process. They used data processing by counting the number of tiles within unit squares and extended results across the mosaic surface through generalisation. They engaged in pattern recognition by detecting repeating areas that allowed for calculation simplification. Reflective abstraction informed their choice to privilege trapezium calculations over rectangles, minimising perspective distortions. Their reasoning was organised through decomposition into sub-steps and by employing algorithm design to manage the operational flow. Generalisation crystallised in their ability to formalise a mathematical model expressed in variables such as a, b, and h.

During interpreting, the participants combined empirical abstraction and decomposition in the assumption that areas included and excluded in a parallelogram-based calculation would balance out, echoing the principles of geometric proof. The participants relied upon data processing by consulting external museum sources to validate assumptions, while debugging guided their decision to retain trapezium-based reasoning as the most robust solution.

Finally, in validating, the participants extended their reasoning through generalisation, testing whether or not their model could be applied to other mosaics from the Antakya museum. Through debugging, they confirmed that trapezium-based calculations could also accommodate rectangular configurations, ensuring inclusivity and robustness of the model. This validation phase highlighted the dual necessity of transferability and corrective reasoning as a means to aligning mathematical abstraction with authentic real-world structures.

The participants showed improvement in solving mathematical modelling problems and also in creating more complex modelling problems. They also showed improvement in expressing actions that occurred in their minds during the problem-solving process. All of these cognitive processes were expressed at the end of the problem (see Fig. 12).

Fig. 12 visualises the overlaps between the participants' identified cognitive actions during the modelling phases and the corresponding CT components. Across the modelling phases, the participants orchestrated CT components to transform a visual estimation task into a transferable mathematical model. Empirical and reflective abstraction structured the problem, decomposition and algorithm design provided systematic scaffolding, while pattern recognition and data processing supported mathematising. Debugging and generalisation were crucial for refining and validating the solution, illustrating how CT practices collectively enabled the construction of a reliable model applicable beyond the initial task.

The current study revealed the dynamic interplay between cognitive processes and CT components within mathematical modelling practices. The findings revealed that as cognitive engagement deepened during modelling tasks, the application of CT components such as abstraction, algorithm design, and debugging intensified correspondingly. While direct one-to-one mapping of modelling phases to CT components proved challenging initially, the participants progressively internalised CT terminology, as evidenced by meta-cognitive statements like *'Let's not skip the algorithm component'* and *'We have made an abstraction here'* in later practices. This integration not only enriched the cognitive processes inherent to modelling but also provided a structured roadmap for navigating complex problem-solving scenarios.

Across all of the sessions, the participants engaged in both posing and solving modelling problems, with data processing emerging as a strategic facilitator in designing systematic algorithms. For instance, during the simplifying phase (see Tables 3 and 5), the participants streamlined variables through empirical abstraction (e.g., selecting the trapezium as a generalisable geometric unit: see Fig. 11c), while the mathematising phase (see Table 4) required reflective abstraction to align real-world variables with mathematical constructs. Transitions between phases (see Figs. 5, 7, and 9) further underscored the role of debugging in error anticipation and

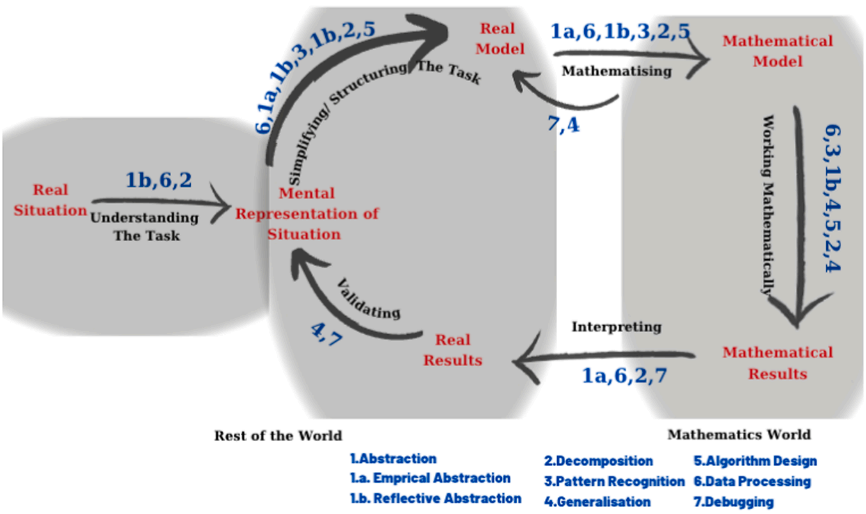

Fig. 12. CT components that emerged during the Mosaic Problem.

iterative refinement. The participants demonstrated that efficient data processing enhanced their capacity to design comprehensive algorithms and manage debugging effectively, highlighting the symbiotic relationship between CT proficiency and modelling success.

These findings underscore the pedagogical imperative of integrating CT into mathematical modelling instruction, positioning CT not as a stabilising element but as a cognitive scaffold employed to amplify problem-solving rigor and metacognitive awareness.

5. Discussion

The current study investigated the overlap between CT and mathematical modelling within the context of a K-12 teachers' professional development course. The findings showed CT integration in mathematics education as emerging from its synergistic relationship with mathematical modelling, framed within an interdisciplinary pedagogical framework. The development and implementation of a specialised TDC in the current research revealed a critical convergence: where teachers' cognitive actions during the modelling process inherently aligned with CT components. These findings carry significant implications that enrich instructional practices, the modelling process, and CT, and that may help to redefine TDCs.

The study implemented a three-module TDC consisting of three practice sessions involving four teachers from diverse academic and professional backgrounds. The participants first learnt about CT and its components in Module 1, then were engaged in mathematical modelling and problem-solving in Module 2, followed by integrating CT components into modelling problems during Module 3. Through iterative sessions (Practices 1–3), the participant teachers reinforced their ability to integrate CT into both problem posing and problem solving. Visual-supported modelling tasks, repeated problem-solving experiences, and reflective abstraction helped enhance the participants' implementation and internalisation of CT components. This experience helped underscore the critical role of iterative practice in deepening cognitive engagement and aligns well with the literature in emphasising that teachers must first experience modelling processes themselves in order to effectively teach them (Blum et al., 2007). The current study also highlighted the direct influence that teachers' own academic backgrounds can have on their understanding and application of CT components (Yadav et al., 2017). The structured TDC structure of the current study, which prioritised experiential learning in modelling prior to advancing on to CT integration, accommodated interdisciplinary profiles and mitigated challenges associated with cross-disciplinary skills transfer. This attribute of the study aligns with prior research which indicated that interdisciplinary integration demands intensive planning and time (Rich et al., 2019). By engaging teachers in hands-on modules and practices, the TDC was able to facilitate active CT integration, mirroring findings that repeated practice can help overcome barriers in interdisciplinary contexts. For instance, the participant teachers' progression from initially being modelling novices to more confident integrators of CT components reflects the transformative potential of structured, practice-driven professional development.

The emphasis placed on visual tools and iterative problem-posing in the current study resonates with broader pedagogical strategies that leverage multimodal approaches to enhance conceptual understanding (Weintrop et al., 2016). Furthermore, the alignment of teachers' cognitive actions (e.g., problem simplification, mathematisation, validation) with CT components (e.g., decomposition, algorithm design, debugging) suggests a natural synergy between modelling and CT, reinforcing their epistemological congruence (Ang, 2021). Future research in this area could explore the longitudinal impact of such integration on student outcomes and the scalability of TDC frameworks across diverse educational contexts.

The current study employed a DBR framework, enabling the participant teachers to progressively recognise CT components through modules and practice sessions. In Practice 1, the participants employed pattern recognition, reflective abstraction, and decomposition to structure the modelling process, followed by algorithm design to solve problems. In Practice 2, data processing and debugging were used to develop new modelling perspectives, while Practice 3 saw the teachers effectively apply each of the CT components. These findings highlight DBR's efficacy in fostering cognitive growth through iterative, practice-oriented design (Killen et al., 2023). The dynamic alignment of instructional design with researcher insight and structured modules helped support cognitive deepening and process-oriented learning outcomes (Mumcu et al., 2022; Shute et al., 2017).

In the current study, as the modelling complexity increased, the participants were seen to leverage CT components more effectively, with overlapping cognitive actions and CT integration having become more pronounced. For instance, in Module 3′s Straw Bale problem, the participant teachers used the human body as a pattern unit. Similarly, in Practice 1, they employed pattern recognition (i.e., average adult male human height in Türkiye) and reflective abstraction (proportional reasoning) to relate building height to this human scale, followed by algorithmic design and decomposition in order to solve the problem. This interplay between modelling and CT components demonstrated their mutual reinforcement, corroborating research by Lv et al. (2023) who found that CT integration can deepen mathematical modelling.

Analysis in the current study revealed that the participant teachers employed algorithm design while formulating and solving modelling problems, leveraging data processing as a strategic tool for systematic planning and solution development. In algorithm design, systematic data organisation and solution structuring operate synergistically, where efficient data processing simultaneously supports algorithm development and strengthens debugging capabilities (Grover & Pea, 2013). The participants demonstrated proficient data processing through the design of more robust algorithms and managed debugging more effectively. Within modelling, error identification through data analysis and solution systematisation was seen to function as a cognitive control mechanism, with combined data processing and debugging having reinforced problem solving (Guzdial, 2016). Further analyses (see Tables 4–6 and Figs. 5, 7, 9) highlighted the supportive role of data processing and debugging across problem-solving stages—simplifying, mathematising, and interpreting—aligning with CT's purported mediating role in interdisciplinary learning (Papert, 1980). The synergistic interplay of data processing, debugging, and algorithm design (Shute et al., 2017) was shown to enrich the participants' cognitive performance, particularly within the modelling cycle.

Collaboration between the participant mathematics teachers and their computer science teacher colleague was shown to significantly enrich CT integration, with interdisciplinary engagement having enhanced the participants' modelling and CT competencies. In Practice 1, the CST participant proposed a modelling task incorporating debugging which was positively received. In Practice 2, the CST sourced a metal railing's length online, adopting it as a unit measure, thereby contextualising CT within daily life. Whereas the three participant mathematics teachers prioritised mathematical processes, the CST was seen to emphasise algorithmic design and debugging. This synergy mirrors CT's interdisciplinary nature (Weintrop et al., 2016; Wing, 2006) and underscores the necessity of cross-disciplinary collaboration for meaningful CT integration (Kallia et al., 2021). Conversely, inadequate or ineffective CT integration in TDCs risks fragmented learning and subsequent limitations in real-world problem-solving (Mumcu et al., 2022). However, the current study demonstrated enriched CT integration through CST-MT collaboration.

The current study can be said to advance the literature in three ways. First, it incorporated CT components into mathematical modelling processes, offering a structured framework through which to analyse how CT functions within real-world modelling situations. Second, the study presented a modelling perspective that connects theory and practice in teacher education, providing a replicable approach for integrating CT into professional development programmes. Third, it empirically identified shared cognitive actions across the different modelling phases, demonstrating how teachers' computational and mathematical thinking can develop together during problem solving. Collectively, these contributions enhance both theoretical and practical understanding of modelling and offer actionable insights for creating future teacher development courses aimed at promoting interdisciplinary cognitive engagement.

5.1. *Limitations*

As with all research, the current study was subject to several limitations. In using DBR within an interdisciplinary TDC, the study examined CT's overlap with mathematical modelling, mediated by CST's role in CS concept acquisition. The study was conducted in Türkiye with a sample consisting of three serving mathematics teachers and one computer science teacher, thereby restricting the representativeness and cultural generalisability of the findings. Implemented within an online environment and relying largely upon video-recorded evidence in addition to researcher observation and group discussion, the study was unable to fully capture all individual cognitive processes. Moreover, the modular sequence and the set tasks were designed specifically for the study's context, and as such their effectiveness may differ within other group or task settings. Employing qualitative methodology with a small sample focused on visual-based problems also reflects common limitations in CT-Mathematics research (Lv et al., 2023), with the absence of mixed methodology and control groups having constrained the generalisability of the study's findings.

The empirical findings of the current study indicate that CS Unplugged activities should not be viewed simply as supplementary tools within mathematics education, but rather as integral components that reinforce the discipline's inherent systematic and creative modes of thought. Future research could therefore aim to extend the current study's findings by examining the intersections of computational thinking and mathematical modelling across varied contexts, incorporating data-driven tasks and technology-enhanced environments. Future studies could expand this design through classroom-embedded implementations that track teachers' and students' interactions during modelling tasks. Scaling up the current TDC to include larger and more diverse teacher cohorts, supported by mixed-method data collection, would help validate the framework's applicability across broader educational contexts and provide richer evidence of its impact upon classroom practices. With an expanded sample size and employing rigorous experimental designs and the utilisation of mixed methods, future research could aim to address the current study's limitations, while investigating emergent CT components in digitally enriched settings may help to further illuminate their interrelations and pedagogical potential.

5.2. *Conclusion*

The current study contributes to the growing body of research on integrating CT into mathematical modelling by illustrating how teachers' cognitive actions align with CT components throughout the modelling process. Based on the study's findings, several practice-oriented implications can be drawn:

- Embedding CT components within modelling practices can enable teachers to utilise CT as a natural part of their thinking processes and supports them in developing systematic solutions to complex problems.
- Iterative and visually supported modelling tasks strengthen teachers' CT skills, including abstraction, decomposition, pattern recognition, generalisation, algorithm design, data processing, and debugging. As these CT components evolve, they begin to operate in an interconnected and mutually reinforcing manner, indicating that CT components and modelling processes feed into one another, with modelling enriching CT and, in turn, CT enhancing modelling.
- Interdisciplinary collaboration between mathematics and computer science teachers can enhance the level of both conceptual and procedural integration.
- Design-based and practice-oriented professional development enhances teachers' CT–modelling interplay.

5.3. *Declaration of generative AI and AI-assisted technology usage*

During the preparation of this work, the authors used Grammarly in order to check the grammar of the writing. After having used this tool, the authors reviewed and edited the content as required and take full responsibility for the content of the publication.

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CRediT authorship contribution statement

Behiye Dinçer Aksoy: Conceptualization, Formal analysis, Writing – review & editing, Writing – original draft, Visualization, Resources, Methodology, Data curation. **Filiz Kuşkaya Mumcu:** Conceptualization, Methodology, Visualization, Validation, Writing – review & editing, Supervision. **Berna Cantürk Günhan:** Conceptualization, Methodology, Visualization, Validation, Supervision, Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no competing interests.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.tsc.2025.102049.

Data availability

The datasets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.

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